ABSOLUTELY CONTINUOUS AND SINGULAR SPECTRAL SHIFT FUNCTIONS

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ABSTRACT. Given a self-adjoint operator H_0, a self-adjoint trace-class operator V and a fixed Hilbert-Schmidt operator F with trivial kernel and co-kernel, using the limiting absorption principle an explicit set of full Lebesgue measure $\Lambda(H_0; F) \subset \mathbb{R}$ is defined, such that for all points λ of the set $\Lambda(H_0 + rV; F) \cap \Lambda(H_0; F)$, where $r \in \mathbb{R}$, the wave $w_\pm(\lambda; H_0 + rV, H_0)$ and the scattering matrices $S(\lambda; H_0 + rV, H_0)$ can be defined unambiguously. Many well-known properties of the wave and scattering matrices and operators are proved, including the stationary formula for the scattering matrix. This version of abstract scattering theory allows, in particular, to prove that

$$\det S(\lambda; H_0 + V, H_0) = e^{-2\pi i \xi^{(a)}(\lambda)}, \quad \text{a.e. } \lambda \in \mathbb{R},$$

where $\xi^{(a)}(\lambda) = \xi^{(a)}_{H_0+V,H_0}(\lambda)$ is the so called absolutely continuous part of the spectral shift function defined by

$$\xi^{(a)}_{H_0+V,H_0}(\lambda) := \frac{d}{d\lambda} \int_0^1 \operatorname{Tr}(V E^{(a)}_{H_0+rV}(\lambda))\, dr$$

and where $E^{(a)}_H(\lambda) = E^{(a)}_{(-\infty,\lambda)}(H)$ denotes the absolutely continuous part of the spectral projection. Combined with the Birman-Krein formula, this implies that the singular part of the spectral shift function

$$\xi^{(s)}_{H_0+V,H_0}(\lambda) := \frac{d}{d\lambda} \int_0^1 \operatorname{Tr}(V E^{(s)}_{H_0+rV}(\lambda))\, dr$$

is an almost everywhere integer-valued function, where $E^{(s)}_H(\lambda) = E^{(s)}_{(-\infty,\lambda)}(H)$ denotes the singular part of the spectral projection.

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Contents

1. Introduction

1.1. **Short summary.** In this paper a new approach is given to abstract scattering theory. This approach is constructive and allows to prove new results in perturbation theory of continuous spectra of self-adjoint operators which the conventional scattering theory is not able to achieve.

Among the results of this paper are: for trace-class perturbations of arbitrary self-adjoint operators:

• A new approach to the spectral theorem of self-adjoint operators (without singular continuous spectrum) via a special constructive representation of the absolutely continuous part (with respect to a fixed self-adjoint operator) of the Hilbert space as a direct integral of fiber Hilbert spaces.

• A new and constructive proof of existence of the wave matrices and of the wave operators.

• A new proof of the multiplicativity property of the wave matrices and of the wave operators.

• A new and constructive proof of the existence of the scattering matrix and of the scattering operator.

• A new proof of the stationary formula for the scattering matrix.

• A new proof of the Kato-Rosenblum theorem.

This paper does not contain only new proofs of existing theorems.

• A new formula (to the best knowledge of the author) for the scattering matrix in terms of chronological exponential.

The main result of this paper is the following

Theorem. Let H_0 be a self-adjoint operator and let V be a trace-class self-adjoint operator in a Hilbert space $\mathcal{H}$. Define a generalized function

$$\xi^{(s)}(\varphi) = \int_0^1 \operatorname{Tr}\left(V\varphi(H_r^{(s)})\right)\,dr, \quad \varphi \in C_c^\infty(\mathbb{R}),$$

where $H_r := H_0 + rV$, and $H_r^{(s)}$ is the singular part of the self-adjoint operator H_r. Then $\xi^{(s)}$ is an absolutely continuous measure and its density $\xi^{(s)}(\lambda)$ (denoted by the same symbol!) is a.e. integer-valued.

Note that in the case of operators with compact resolvent this theorem is well known, and the function $\xi^{(s)}(\lambda)$ in this case coincides with spectral flow [APS, APS$_2$, Ge, Ph, Ph$_2$, CP, CP$_2$, ACDS, ACS, Az$_4$]. Spectral flow is integer-valued just by definition as a total Fredholm index of a path of operators. In the case of operators with compact resolvent instead of $H_r^{(s)}$ one writes H_r, since in this case the continuous spectrum is absent, so that $H_r^{(s)} + H_r$.

The above theorem strongly suggests that the function $\xi^{(s)}(\lambda)$, which I call the singular part of the spectral shift function, calculates the spectral flow of the singular spectrum even in the presence and inside of the absolutely continuous spectrum.

Finally, it is worth to stress that the new approach to abstract scattering theory given in this paper has been invented with the sole purpose to prove the above theorem. Existing versions of scattering theory turned out to be insufficient for this purpose. At the same time, this approach seems to have a value of its own. In particular, properly adjusted, it allows to unify the trace-class and smooth scattering theories, thus solving a long-standing problem mentioned in the introduction of D. Yafaev's book [Y].

1.2. **Introduction.** Let H_0 be a self-adjoint operator, V be a self-adjoint trace-class operator and let $H_1 = H_0 + V$. The Lifshits-Kreĭn spectral-shift function [L, Kr] is the unique L^1-function $\xi(\cdot) = \xi_{H_1,H_0}(\cdot)$, such that for all $f \in C_c^\infty(\mathbb{R})$ the trace formula

$$\operatorname{Tr}(f(H_1) - f(H_0)) = \int f'(\lambda)\xi_{H_1,H_0}(\lambda)\,d\lambda$$

holds. The Birman-Solomyak formula for the spectral shift function [BS$_2$] asserts that

$$\xi_{H_1,H_0}(\lambda) = \frac{d}{d\lambda}\int_0^1 \operatorname{Tr}(VE_\lambda^{H_r})\,dr, \quad \text{a.e. } \lambda, \tag{1}$$

where

$$H_r = H_0 + rV,$$

and $E_\lambda^{H_r}$ is the spectral resolution of H_r. This formula was established by V. A. Javrjan in [J] in the case of perturbations of the boundary condition of a Sturm-Liouville operator on $[0,\infty)$, which corresponds to rank-one perturbation of H_0. The Birman-Solomyak formula is also called the spectral averaging formula. A simple proof of this formula was found in [S$_2$]. There is enormous literature on the subject of spectral averaging, cf. e.g. [GM$_2$, GM, Ko] and references therein. A survey on the spectral shift function can be found in [BP].

Let $S(\lambda; H_1, H_0)$ be the scattering matrix of the pair $H_0, H_1 = H_0 + V$ (cf. [BE], see also [Y]). In [BK] M. Sh. Birman and M. G. Kreĭn established the following formula

$$\det S(\lambda; H_1, H_0) = e^{-2\pi i \xi(\lambda)} \quad \text{a.e. } \lambda \in \mathbb{R} \tag{2}$$

for trace-class perturbations $V = H_1 - H_0$ and arbitrary self-adjoint operators H_0. This formula is a generalization of a similar result of V. S. Buslaev and L. D. Faddeev [BF] for Sturm-Liouville operators on $[0,\infty)$.

In [Az] I introduced the absolutely continuous and singular spectral shift functions by the formulae

$$\xi^{(a)}_{H_1,H_0}(\lambda) = \frac{d}{d\lambda}\int_0^1 \mathrm{Tr}\left(VE^{H_r}_\lambda P^{(a)}(H_r)\right)\,dr, \quad \text{a.e. } \lambda, \tag{3}$$

$$\xi^{(s)}_{H_1,H_0}(\lambda) = \frac{d}{d\lambda}\int_0^1 \mathrm{Tr}\left(VE^{H_r}_\lambda P^{(s)}(H_r)\right)\,dr, \quad \text{a.e. } \lambda, \tag{4}$$

where $P^{(a)}(H_r)$ (respectively, $P^{(s)}(H_r)$) is the projection onto the absolutely continuous (respectively, singular) subspace of H_r. These formulae are obvious modifications of the Birman-Solomyak spectral averaging formula, and one can see that

$$\xi = \xi^{(s)} + \xi^{(a)}.$$

In [Az] it was observed that for n-dimensional Schrödinger operators $H_r = -\Delta + rV$ with quickly decreasing potentials V the scattering matrix $S(\lambda; H_r, H_0)$ is a continuous operator-valued function of r and it was shown that

$$-2\pi i\xi^{(a)}_{H_r,H_0}(\lambda) = \log\det S(\lambda; H_r, H_0), \tag{5}$$

where the logarithm is defined in such a way that the function

$$[0,r] \ni s \mapsto \log\det S(\lambda; H_s, H_0)$$

is continuous. It was natural to conjecture that some variant of this formula should hold in general case. In particular, this formula, compared with the Birman-Krein formula (2), has naturally led to a conjecture that the singular part of the spectral shift function is an a.e. integer-valued function. In case of n-dimensional Schrödinger operators with quickly decreasing potentials this is a well-known result, since these operators do not have singular spectrum on the positive semi-axis. In [Az$_2$] it was observed that even in the case of operators which admit embedded eigenvalues the singular part of the spectral shift function is also either equal to zero on the positive semi-axis or in any case it is integer-valued.

In this paper I give a positive solution of this conjecture for trace-class perturbations of arbitrary self-adjoint operators.

The proof of (5) is based on the following formula for the scattering matrix

$$S(\lambda; H_r, H_0) = \mathrm{Texp}\left(-2\pi i\int_0^r w_+(\lambda; H_0, H_s)\Pi_{H_s}(V)(\lambda)w_+(\lambda; H_s, H_0)\,ds\right), \tag{6}$$

where $\Pi_{H_s}(V)(\lambda)$ is the so-called infinitesimal scattering matrix (see (88)). If λ is fixed, then for this formula to make sense, the wave matrix $w_+(\lambda; H_s, H_0)$ has to be defined for all $s \in [0, r]$, except possibly a discrete set. In the case of Schrödinger operators

$$H = -\Delta + V$$

in $\mathbb{R}^n$ with a short range potentials (in the sense of [Ag]), the wave matrices $w_\pm(\lambda; H_s, H_0)$ are well-defined, since there are explicit formulae for them, cf. e.g. [Ag, BY$_2$, Ku, Ku$_2$, Ku$_3$]. For example, if λ does not belong to the discrete set $e_+(H)$ of embedded eigenvalues

of H, then the scattering matrix $S(\lambda)$ exists as an operator from $L^2(\Sigma)$ to $L^2(\Sigma)$, where $\Sigma = \{\omega \in \mathbb{R}^n \colon\ |\omega| = 1\}$ (cf. e.g. [Ag, Theorem 7.2]).

The situation is quite different in the case of the main setting of abstract scattering theory [BW, BE, RS$_3$, Y], which considers trace-class perturbations of arbitrary self-adjoint operators. A careful reading of proofs in [BE, Y] shows that one takes an arbitrary core of the spectrum of the initial operator H_0 and during the proofs one throws away from a core of the spectrum several finite and even countable families of null sets. Furthermore, the nature of the initial core of the spectrum and the nature of the null sets being thrown away are not clarified. They depend on arbitrarily chosen objects. This is in sharp contrast with potential scattering theory, where non-existence of the wave matrix or the scattering matrix at some point λ of the absolutely continuous spectrum means that λ is an embedded eigenvalue, cf. e.g. [Ag].

So, in the case of trace-class perturbations of arbitrary self-adjoint operators, given a fixed λ (from some predefined full set Λ) the existence of the wave matrix for all $r \in [0,1]$, except possibly a discrete set, cannot be established by usual means. In order to make the argument of the proof of (6), given in [Az], work for trace-class (to begin with) perturbations of arbitrary self-adjoint operators, one at least needs to give an explicit set of full measure Λ, such that for all λ from Λ all the necessary ingredients of the scattering theory, such as $w_\pm(\lambda; H_r, H_0)$, $S(\lambda; H_r, H_0)$ and $Z(\lambda; G)$ exist. One of the difficulties here is that the spectrum of an arbitrary self-adjoint operator, unlike the spectrum of Schrödinger operators, can be very bad: it can, say, have everywhere dense pure point spectrum, or a singular continuous spectrum, or even both.

To the best knowledge of the author, abstract scattering theory in its present form (cf. [BW, BE, RS$_3$, Y]) does not allow to resolve this problem. In the present paper a new abstract scattering theory is developed (to the best knowledge of the author).

In this theory, given a self-adjoint operator H_0 on a Hilbert space $\mathcal{H}$ with the so-called frame F and a trace-class perturbation V, an explicit set of full measure $\Lambda(H_0; F)$ is defined in a canonical (constructive) way via the data (H_0, F), such that for all $\lambda \in \Lambda(H_0; F) \cap \Lambda(H_r; F)$ the wave matrices $w_\pm(\lambda; H_r, H_0)$ exist, and moreover, explicitly constructed.

Definition 1.1. *A* frame F *in a Hilbert space* $\mathcal{H}$ *is a sequence*

$$((\varphi_1, \kappa_1), (\varphi_2, \kappa_2), (\varphi_3, \kappa_3), \ldots,),$$

where $(\kappa_j)_{j=1}^\infty$ *is an* ℓ_2*-sequence of positive numbers, and* $(\varphi_j)_{j=1}^\infty$ *is an orthonormal basis of* $\mathcal{H}$.

It is convenient to encode the information about a frame in a Hilbert-Schmidt operator with trivial kernel and co-kernel

$$F \colon \mathcal{H} \to \mathcal{K}, \quad F = \sum_{j=1}^\infty \kappa_j \langle \varphi_j, \cdot \rangle \, \psi_j,$$

where $\mathcal{K}$ is another Hilbert space and $(\psi_j)_{j=1}^\infty$ is an orthonormal basis in $\mathcal{K}$. The nature of the Hilbert space $\mathcal{K}$ and of the basis $(\psi_j)_{j=1}^\infty$ is immaterial, so that one can actually take $\mathcal{K} = \mathcal{H}$ and $(\psi_j)_{j=1}^\infty = (\varphi_j)_{j=1}^\infty$.

Once a frame (operator) F is fixed in $\mathcal{H}$, given a self-adjoint operator H_0 on $\mathcal{H}$, the frame enables to construct explicitly:

(1) a standard set of full measure $\Lambda(H_0; F)$;
(2) for every $\lambda \in \Lambda(H_0; F)$, an explicit (to be fiber) Hilbert space $\mathfrak{h}_\lambda \subset \ell_2$;
(3) a measurability base $\{\varphi_j(\cdot)\}$, $j = 1, 2, \ldots$, where all functions $\varphi_j(\lambda) \in \mathfrak{h}_\lambda, j = 1, 2, \ldots$, are explicitly defined for *all* $\lambda \in \Lambda(H_0; F)$;
(4) (as a consequence) a direct integral of Hilbert spaces
$$\mathcal{H} := \int_{\Lambda(H_0;F)}^{\oplus} \mathfrak{h}_\lambda \, d\lambda,$$
where the case of $\dim \mathfrak{h}_\lambda = 0$ is *not excluded*.
(5) Further, considered as a rigging, a frame F generates a triple of Hilbert spaces $\mathcal{H}_1 \subset \mathcal{H} = \mathcal{H}_0 \subset \mathcal{H}_{-1}$ with scalar products
$$\langle f, g\rangle_{\mathcal{H}_\alpha} = \left\langle |F|^{-\alpha} f, |F|^{-\alpha} g\right\rangle, \quad \alpha = -1, 0, 1$$
and natural isomorphisms
$$\mathcal{H}_{-1} \xrightarrow{|F|} \mathcal{H} \xrightarrow{|F|} \mathcal{H}_1.$$
(6) for any $\lambda \in \Lambda(H_0; F)$ we have an *evaluation* operator
$$\mathcal{E}_\lambda = \mathcal{E}_{\lambda+i0} \colon \mathcal{H}_1 \to \mathfrak{h}_\lambda;$$
$$\mathcal{E} \colon \mathcal{H}_1 \to \mathcal{H}.$$
The operator $\mathcal{E}_\lambda \colon \mathcal{H}_1 \to \mathfrak{h}_\lambda$ is a Hilbert-Schmidt operator, and the operator $\mathcal{E}$, considered as an operator $\mathcal{H} \to \mathcal{H}$, extends continuously to a *unitary isomorphism* of the absolutely continuous part (with respect to H_0) of $\mathcal{H}$ to $\mathcal{H}$, and, moreover, the operator $\mathcal{E}$ *diagonalizes* the absolutely continuous part of H_0.

Here is a quick description of this construction.

Definition 1.2. *A point $\lambda \in \mathbb{R}$ belongs to $\Lambda(H_0; F)$ if and only if*

(i) *the operator $FR_{\lambda+iy}(H_0)F^*$ has a limit in the Hilbert-Schmidt norm as $y \to 0^+$,*
(ii) *the operator $F \operatorname{Im} R_{\lambda+iy}(H_0)F^*$ has a limit in the trace-class norm as $y \to 0^+$.*

It follows from the limiting absorption principle (cf. [B, BE] and [Y, Theorems 6.1.5, 6.1.9]), that $\Lambda(H_0; F)$ has full Lebesgue measure, and that for all $\lambda \in \Lambda(H_0; F)$ the matrix
$$\varphi(\lambda) := (\varphi_{ij}(\lambda)) = \frac{1}{\pi} \left(\kappa_i \kappa_j \left\langle \varphi_i, \operatorname{Im} R_{\lambda+i0}(H_0) \varphi_j\right\rangle\right)$$
exists and is a non-negative trace-class operator on ℓ_2 (Proposition 3.4). The value $\varphi_j(\lambda)$ of the vector φ_j at $\lambda \in \Lambda(H_0; F)$ is defined as the j-th column $\eta_j(\lambda)$ of the Hilbert-Schmidt

operator $\sqrt{\varphi(\lambda)}$ over the weight κ_j of φ_j :

$$\varphi_j(\lambda) = \kappa_j^{-1}\eta_j(\lambda).$$

It is not difficult to see that if $f \in \mathcal{H}_1(F)$, so that

$$f = \sum_{j=1}^{\infty} \kappa_j \beta_j \varphi_j,$$

where $(\beta_j) \in \ell_2$, then the series

$$f(\lambda) := \sum_{j=1}^{\infty} \kappa_j \beta_j \varphi_j(\lambda) = \sum_{j=1}^{\infty} \beta_j \eta_j(\lambda)$$

absolutely converges in ℓ_2. The fiber Hilbert space $\mathfrak{h}_\lambda$ is by definition the closure of the image of $\mathcal{H}_1$ under the map

$$\mathcal{E}_\lambda \colon \mathcal{H}_1 \ni f \mapsto f(\lambda) \in \ell_2.$$

The image of the set of frame vectors φ_j under the map $\mathcal{E}$ form a measurability base of a direct integral of Hilbert spaces

$$\mathfrak{H} := \int_{\Lambda(H_0;F)}^{\oplus} \mathfrak{h}_\lambda \, d\lambda,$$

and the operator

$$\mathcal{E} \colon \mathcal{H}_1 \to \mathfrak{H}$$

is bounded as an operator from $\mathcal{H}$ to $\mathfrak{H}$, vanishes on the singular subspace $\mathcal{H}^{(s)}(H_0)$ of H_0, is isometric on the absolutely continuous subspace $\mathcal{H}^{(a)}(H_0)$ of H_0 with the range $\mathfrak{H}$ (Propositions 4.11, 4.17) and is diagonalizing for H_0 (Theorem 4.19).

* * *

So far, we have had one self-adjoint operator H_0 acting in $\mathcal{H}$. Let V be a self-adjoint trace-class operator. Let a frame F be such that $V = F^*JF$, where $J \colon \mathcal{K} \to \mathcal{K}$ is a self-adjoint bounded operator. Clearly, for any trace-class operator such a frame exists. This means that the operator V can be considered as a bounded operator

$$V \colon \mathcal{H}_{-1} \to \mathcal{H}_1.$$

By definition, if $\lambda \in \Lambda(H_0; F)$, then the limit

$$R_{\lambda+i0}(H_0) \colon \mathcal{H}_1 \to \mathcal{H}_{-1}$$

exists in the Hilbert-Schmidt norm and the limit

$$\operatorname{Im} R_{\lambda+i0}(H_0) \colon \mathcal{H}_1 \to \mathcal{H}_{-1}$$

exists in the trace-class norm. So, if $\lambda \in \Lambda(H_0; F) \cap \Lambda(H_1; F)$, then the following operator is a well-defined trace-class operator (from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$)

$$\mathfrak{a}_\pm(\lambda; H_1, H_0) := \Big[1 - R_{\lambda\mp i0}(H_1)V\Big] \cdot \frac{1}{\pi} \operatorname{Im} R_{\lambda+i0}(H_0).$$

Let $\lambda \in \Lambda(H_0; F) \cap \Lambda(H_1; F)$, where $H_1 = H_0 + V$, so that both fiber Hilbert spaces $\mathfrak{h}_\lambda^{(0)}$ and $\mathfrak{h}_\lambda^{(1)}$ are well-defined. Then there exists a unique (for each sign $\pm$) operator

$$w_\pm(\lambda; H_1, H_0) \colon \mathfrak{h}_\lambda^{(0)} \to \mathfrak{h}_\lambda^{(1)}$$

such that for any $f, g \in \mathcal{H}_1$ the equality

$$\langle \mathcal{E}_\lambda(H_1) f, w_\pm(\lambda; H_1, H_0) \mathcal{E}_\lambda(H_0) g \rangle = \langle f, \mathfrak{a}_\pm(\lambda; H_1, H_0) g \rangle_{1,-1}$$

holds, where $\langle \cdot, \cdot \rangle_{1,-1}$ is the pairing of the rigged Hilbert space $(\mathcal{H}_1, \mathcal{H}, \mathcal{H}_{-1})$. The operator

$$w_\pm(\lambda; H_1, H_0)$$

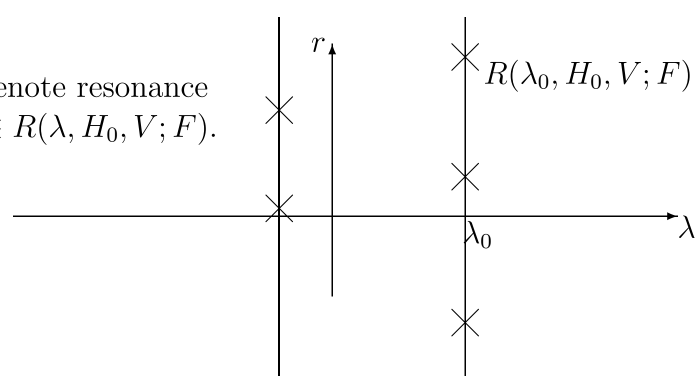
is correctly defined, and, moreover, it is unitary and satisfies multiplicative property. The operator $w_\pm(\lambda; H_1, H_0)$ is actually the wave matrix, which is thus explicitly constructed for *all* λ from the set of full Lebesgue measure $\Lambda(H_0; F) \cap \Lambda(H_1; F)$.

So far we considered a pair of operators H_0 and H_1. But if the aim is to prove the formula (6), then one needs to make sure that the wave matrix $w_\pm(\lambda; H_r, H_0)$ exists for all, with a possible exception of some small set, values of $r \in [0, 1]$. It turns out that, indeed, the wave matrix $w_\pm(\lambda; H_r, H_0)$ is defined for all r except a discrete set, as follows from the following simple but important property of the set $\Lambda(H_0; F)$ (Theorem 5.8):

$$\text{if } \ \lambda \in \Lambda(H_0; F), \ \ \text{then } \lambda \in \Lambda(H_r; F) \ \text{ for all } r \notin R(\lambda, H_0, V; F),$$

where $R(\lambda, H_0, V; F)$ is a discrete set of special importance called resonance set (see the picture below).

Crosses denote resonance points $r \in R(\lambda, H_0, V; F)$.



If λ is an eigenvalue of $H_r = H_0 + rV$, then $r \in R(\lambda, H_0, V; F)$ for any F. But $R(\lambda, H_0, V; F)$ may contain other points as well, which may depend on F. This partly justifies the terminology "resonance points" and gives a basis for classification of resonance points into two different types.

So, the set $\{(\lambda, r) \colon \lambda \in \Lambda(H_r; F)\}$ behaves very regularly with respect to r, but it does not do so with respect to λ : while for fixed $r_0 \in \mathbb{R}$ and $\lambda_0 \in \mathbb{R}$

the set $\{\lambda \in \mathbb{R} \colon \lambda \notin \Lambda(H_{r_0}; F)\}$ can be a more or less arbitrary null set, the set $\{r \in \mathbb{R} \colon \lambda_0 \notin \Lambda(H_r; F)\}$ is a discrete set, i.e. a set with no finite accumulation points.

Further, the multiplicative property of the wave matrix

$$w_\pm(\lambda; H_{r_2}, H_{r_0}) = w_\pm(\lambda; H_{r_2}, H_{r_1}) w_\pm(\lambda; H_{r_1}, H_{r_0})$$

is proved (Theorem 6.16), where r_2, r_1, r_0 do not belong to the above mentioned discrete resonance set $R(\lambda, H_0, V; F)$. As is known (cf. [Y, Subsection 2.7.3]), the proof of this property for the wave operator $W_\pm(H_1, H_0)$ composes the main difficulty of the stationary approach to the abstract scattering theory. A bulk of this paper is devoted to the definition of $w_\pm(\lambda; H_r, H_0)$ for all $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$ and the proof of the multiplicative property. This is the main feature of the new scattering theory given in this paper. Further, for all $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$ the scattering matrix $S(\lambda; H_r, H_0)$ is *defined* as an operator from $\mathfrak{h}_\lambda$ to $\mathfrak{h}_\lambda$ by the formula

$$S(\lambda; H_r, H_0) = w_+^*(\lambda; H_r, H_0) w_-(\lambda; H_r, H_0).$$

The scattering operator $\mathbf{S}(H_r, H_0) \colon \mathcal{H}^{(a)}(H_0) \to \mathcal{H}^{(a)}(H_0)$ is defined as the direct integral of scattering matrices:

$$\mathbf{S}(H_r, H_0) := \int_{\Lambda(H_0;F)\cap\Lambda(H_r;F)} S(\lambda; H_r, H_0)\, d\lambda.$$

For all $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$ the stationary formula for the scattering matrix

$$S(\lambda; H_r, H_0) = 1_\lambda - 2\pi i \mathcal{E}_\lambda r V (1 + r R_{\lambda + i0}(H_0) V)^{-1} \mathcal{E}_\lambda^\diamondsuit$$

is proved (Theorem 8.5). Though the scattering matrix $S(\lambda; H_r, H_0)$ does not exist for resonance points $r \in R(\lambda, H_0, V; F)$, a simple but important property of the scattering matrix is that it admits analytic continuation to the resonance points (Proposition 8.8). The stationary formula enables to show that for all $\lambda \in \Lambda(H_0; F)$ and all r not in the resonance set $R(\lambda, H_0, V; F)$, the formula (6) holds (Theorem 8.11), where for all non-resonance points r the infinitesimal scattering matrix is defined as

$$\Pi_{H_r}(V)(\lambda) = \mathcal{E}_\lambda(H_r) V \mathcal{E}_\lambda^\diamondsuit(H_r) \colon \mathfrak{h}_\lambda^{(r)} \to \mathfrak{h}_\lambda^{(r)}$$

and where $\mathcal{E}_\lambda^\diamondsuit = |F|^{-2}\, \mathcal{E}_\lambda^*$.

The main object of the abstract scattering theory given in [BE, Y], the wave operator $W_\pm(H_r, H_0)$, is defined as the direct integral of the wave matrices

$$W_\pm(H_r, H_0) = \int^\oplus_{\Lambda(H_r;F)\cap\Lambda(H_0;F)} w_\pm(\lambda; H_r, H_0)\, d\lambda.$$

The usual definition

$$W_\pm(H_r, H_0) = \text{s-}\lim_{t \to \pm\infty} e^{itH_r} e^{-itH_0} P_0^{(a)}$$

of the wave operator becomes a theorem (Theorem 7.4). The formula

$$\mathbf{S}(H_r, H_0) = W_+^*(H_r, H_0) W_-(H_r, H_0),$$

which is usually considered as definition of the scattering matrix, obviously holds.

This new scattering theory has allowed to prove (5) for all $\lambda \in \Lambda(H_0; F)$ (Theorem 9.8). Combined with the Birman-Krein formula (2) this implies that the singular part of the spectral shift function is an a.e. integer-valued function for arbitrary trace-class perturbations of arbitrary self-adjoint operators (Theorem 9.11):

$$\xi^{(s)}_{H_1,H_0}(\lambda) \in \mathbb{Z} \ \text{ for a.e.} \quad \lambda \in \mathbb{R}.$$

Theorem 9.11 is the main result of this paper. This result is to be considered as unexpected, since the definition (4) of the singular part of the spectral shift function does not suggest anything like this.

I would like to stress that even though the scattering theory presented in this paper is different in its nature from the conventional scattering theory given in [BE, Y], many essential ideas are taken from [BE, Y] (cf. also [BW, RS_3]), and essentially no new results appear until subsection 8.2, though most of the proofs are original (to the best of the author's knowledge). At the same time, this new approach to abstract scattering theory is simpler than that of given in [Y], and it is this new approach which allows one to prove main results of this paper.

2. Preliminaries

In these preliminaries I follow mainly [GK, RS, S, Y]. Details and (omitted) proofs can be found in these references. A partial purpose of these preliminaries is to fix notation and terminology.

2.1. **Notation.** $\mathbb{R}$ is the set of real numbers. $\mathbb{C}$ is the set of complex numbers. $\mathbb{C}_+$ is the open upper half-plane of the complex plane $\mathbb{C}$.

2.2. **Measure theory.** Here we collect some definitions from measure theory. Details can be found in D. Yafaev's book [Y].

The σ-algebra $B(\mathbb{R})$ of Borel sets is generated by open subsets of $\mathbb{R}$. By a measure on $\mathbb{R}$ we mean a locally-finite non-negative countably additive function m defined on the σ-algebra of Borel sets. Locally-finite means that the measure of any compact set is finite. By a *Borel support* of a measure m we mean any Borel set X whose complement has zero m-measure: $m(\mathbb{R} \setminus X) = 0$. By the *closed support* of a measure m we mean the smallest closed Borel support of m. The closed support exists and is unique. By a *minimal Borel support* we mean a Borel support X such that for any other Borel support X' the equality $m(X' \setminus X) = 0$ holds. Note that the closed support of a measure is not necessarily minimal.

By $|X|$ we denote the Lebesgue measure of a Borel set X. A Borel set Z is called a *null set*, if it has zero Lebesgue measure: $|Z| = 0$. A Borel set Λ is called *full set*, if the complement of X is a null set: $|\mathbb{R} \setminus \Lambda| = 0$. Full sets will usually be denoted by Λ, with indices and arguments, if necessary.

A measure m is called *absolutely continuous*, if for any null set Z the equality $m(Z) = 0$ holds. The Radon-Nikodym theorem asserts that a measure m is absolutely continuous if and only if there exists a locally-summable non-negative function f such that for any Borel set X

$$m(X) = \int_X f(\lambda)\,d\lambda.$$

A measure m is called *singular*, if there exists a null Borel support of m, that is, a Borel support of zero Lebesgue measure. Any measure m admits a unique decomposition

$$m = m^{(a)} + m^{(s)}$$

into the sum of an absolutely continuous measure $m^{(a)}$ and a singular measure $m^{(s)}$.

Two measures m_1 and m_2 have the same spectral type, if they are absolutely continuous with respect to each other, that is, if $m_1(X) = 0$ for some Borel set X, then $m_2(X) = 0$, and vice versa.

The abbreviation a.e. will always refer to the Lebesgue measure.

Two measures are mutually singular, if they have non-intersecting Borel supports.

A signed measure is a locally finite countably-additive function m defined on the σ-algebra of Borel sets. Every signed measure m admits a unique Hahn decomposition:

$$m = m_+ - m_-,$$

where non-negative measures m_- and m_+ are mutually singular. The measure $|m| := m_+ + m_-$ is called total variation of m.

2.2.1. *Vitali's theorem.* Apart of Lebesgue's dominated convergence theorem, we shall use Vitali's theorem twice. This is the following theorem (for a proof see [Nat]).

Theorem 2.1. *Let X be a Borel subset of $\mathbb{R}$. Suppose for functions $f_y \in L^1(\mathbb{R})$, $y > 0$, the integrals*

$$\int_X f_y(\lambda)\,d\lambda$$

tend to zero uniformly with respect to y as $|X| \to 0$. Suppose also the same for $X = (-\infty, N) \cup (N, \infty)$ as $N \to \infty$. If for a.e. $\lambda \in \mathbb{R}$

$$\lim_{y\to 0} f_y(\lambda) = f(\lambda),$$

then the function f is summable and

$$\lim_{y\to 0} \int_{-\infty}^{\infty} f_y(\lambda)\,d\lambda = \int_{-\infty}^{\infty} f(\lambda)\,d\lambda.$$

2.2.2. *Poisson integral.* Let F be a function of bounded variation on $\mathbb{R}$. Poisson integral $\mathcal{P}_F$ of F is the following function of two variables:

$$\mathcal{P}_F(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{dF(t)}{(x-t)^2+y^2}.$$

The function

$$P_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2} \tag{7}$$

is the kernel of the Poisson integral and

$$\mathcal{P}_F(x,y) = P_y * dF(x).$$

The family $\{P_y(x), y>0\}$ form an approximate unit for the delta-function, that is, all these functions are non-negative, an integral of each of the functions P_y is equal to 1 and P_y converge in distributions sense to the Dirac's delta function δ.

In case when F is the distribution function of a summable function $f \in L_1(\mathbb{R})$, allowing an abuse of terminology, we also say that $P_y * f(x)$ is the Poisson integral of the function f.

Lemma 2.2. *Let $g \in L_1(\mathbb{R})$ and let g_y be the Poisson integral of g. If X is a Borel set, then the integral*

$$\int_X |g_y(\lambda)|\, d\lambda$$

converges to zero as $|X| \to 0$, uniformly with respect to $y > 0$.

Proof. Recall that $g_y(\lambda) = P_y * g(\lambda)$, where P_y is the kernel of the Poisson integral (7).

Let $\varepsilon > 0$. There exists $\delta > 0$ such that for any Borel set X with $|X| < \delta$ the inequality

$$\int_X |g(\lambda)|\, d\lambda < \varepsilon$$

holds. Further, if $|X| < \delta$, then for any $y > 0$

$$\begin{aligned}
\int_X |g_y(\lambda)|\, d\lambda &= \int_X \left| \int_{\mathbb{R}} g(\lambda - t) P_y(t)\, dt \right| d\lambda \\
&\leqslant \int_X \left(\int_{\mathbb{R}} |g(\lambda - t)|\, P_y(t)\, dt \right) d\lambda \\
&\leqslant \int_{\mathbb{R}} P_y(t) \left(\int_X |g(\lambda - t)|\, d\lambda \right) dt \\
&\leqslant \int_{\mathbb{R}} P_y(t) \left(\int_{X-t} |g(\lambda)|\, d\lambda \right) dt \\
&< \varepsilon \int_{\mathbb{R}} P_y(t)\, dt = \varepsilon.
\end{aligned}$$

Proof is complete. $\square$

2.2.3. *Fatou's theorem.* For reader's convenience, in this subsection we give the proof of Fatou's theorem. The proof has been taken from [Ho], where Fatou's theorem is proved for the disk in greater generality of non-tangential limit. For a discussion of Fatou's theorem see also [Y].

Theorem 2.3. *Let F be a function of bounded variation on $\mathbb{R}$. If at some point $x_0 \in \mathbb{R}$ the function F has the symmetric derivative*

$$F'_{sym}(x_0) := \lim_{h \to 0^+} \frac{F(x_0+h) - F(x_0-h)}{2h},$$

then the limit of the Poisson integral of F

$$\lim_{y \to 0^+} \mathcal{P}_F(x_0, y)$$

exists and is equal to $F'_{sym}(x_0)$. In particular, the limit exists for a.e. x_0.

Proof. We have

$$\begin{aligned} \mathcal{P}_F(x,y) = P_y * dF(x) &= \int_{-\infty}^{\infty} P_y(x-t) dF(t) \\ &= P_y(x-t)F(t)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} P'_y(x-t)F(t)\, dt \\ &= \int_{-\infty}^{\infty} P'_y(x-t)F(t)\, dt \end{aligned}$$

where the last equality follows, since F is bounded and the Poisson kernel vanishes at $\pm\infty$. Further, since $P'_y(t)$ is an odd function,

$$\begin{aligned} \mathcal{P}_F(x,y) &= \int_{-\infty}^{\infty} P'_y(t)F(x-t)\, dt \\ &= \int_0^{\infty} P'_y(t)F(x-t)\, dt + \int_{-\infty}^{0} P'_y(t)F(x-t)\, dt \\ &= \int_0^{\infty} P'_y(t)\Big[F(x-t) - F(x+t)\Big]\, dt \\ &= \int_0^{\infty} 2K_y(t)\frac{F(x+t)-F(x-t)}{2t}\, dt, \end{aligned}$$

where

$$K_y(t) = -tP'_y(t) = \frac{2yt^2}{\pi(t^2+y^2)^2}$$

is an even function. So,

$$\mathcal{P}_F(x,y) = \int_{-\infty}^{\infty} K_y(t)\frac{F(x+t)-F(x-t)}{2t}\, dt.$$

Further, since

$$\int_{-\infty}^{\infty} K_y(t)\, dt = -\int_{-\infty}^{\infty} tP'_y(t)\, dt = \int_{-\infty}^{\infty} P_y(t)\, dt = 1,$$

the family of functions $\{K_y \colon y > 0\}$ is an approximate identity.

Now, if F has a symmetric derivative at x_0, then the function

$$G(t) = \frac{F(x_0+t) - F(x_0-t)}{2t}$$

is continuous at 0 and its value at 0 is $F'_{sym}(x_0)$. Since $\{K_y\}$ is an approximate identity, and since G is continuous at 0, it follows that

$$\lim_{y\to 0^+} \mathcal{P}_F(x_0,y) = \lim_{y\to 0^+} \int_{-\infty}^{\infty} K_y(t) G(t)\,dt = G(0) = F'_{sym}(x_0).$$

Obviously, if a function is differentiable at some point, then it also has the symmetric derivative at that point. Note that in the above theorem the case of infinite symmetric derivative is not excluded.

Since by Lebesgue's differentiation theorem a function of bounded variation F is a.e. differentiable, it follows that the limit $\lim\limits_{y\to 0^+} \mathcal{P}_F(x,y)$ exists for a.e. $x \in \mathbb{R}$.

The proof is complete. □

2.2.4. *Privalov's theorem.* Let $F \colon \mathbb{R} \to \mathbb{C}$ be a function of bounded variation. The Cauchy-Stieltjes transform of F is a function holomorphic in both the upper and the lower complex half-planes $\mathbb{C}_\pm$; this function is defined by the formula

$$\mathcal{C}_F(z) = \int_{-\infty}^{\infty} (x-z)^{-1}\,dF(x).$$

The following theorem is known as Privalov theorem (cf. [Pr], [Y, Theorem 1.2.5]). This theorem can be formulated for an upper half-plane or, equivalently, for a unit disk. The proof of the theorem can also be found in [AhG, Chapter VI, §59, Theorem 1].

Theorem 2.4. *Let $F \colon \mathbb{R} \to \mathbb{C}$ be a function of bounded variation. The limits values*

$$\mathcal{C}_F(\lambda \pm i0) := \lim_{y\to 0^+} \mathcal{C}_F(\lambda \pm iy)$$

of the Cauchy-Stieltjes transform $\mathcal{C}_F(z)$ of F exist for a.e. $\lambda \in \mathbb{R}$, and for a.e. $\lambda \in \mathbb{R}$ the equality

$$\mathcal{C}_F(\lambda \pm i0) = \pm \pi i \frac{dF(\lambda)}{d\lambda} + \text{p.v.} \int_{-\infty}^{\infty} (\mu - \lambda)^{-1}\,dF(\mu) \tag{8}$$

holds, where the principal value integral on the right-hand side also exists for a.e. $\lambda \in \mathbb{R}$.

Since the imaginary part of the Cauchy-Stieltjes transform of F is the Poisson kernel of F :

$$\frac{1}{\pi} \operatorname{Im} \mathcal{C}_F(\lambda + iy) = \mathcal{P}_F(\lambda, y), \tag{9}$$

the convergence of $\frac{1}{\pi}\operatorname{Im}\mathcal{C}_F(\lambda \pm iy)$ to $\pm F'(\lambda)$ for a.e. λ follows from Fatou's theorem 2.3. The convergence of the real part to the p.v. integral will not be used in this paper, and therefore is omitted.

2.2.5. *The set* $\Lambda(f)$. It is customary to consider a summable function $f \in L^1(\mathbb{R})$ as a class of equivalent functions, where two functions are considered to be equivalent if they coincide everywhere except a null set. So, a summable function is defined up to a set of Lebesgue measure zero. In this way, in general one cannot ask what is the value of a summable function f at, say, $\sqrt{2}$. In this paper we take a different approach. By a summable function we mean a complex-valued summable function f which is *explicitly* defined on some explicit set of full Lebesgue measure.

Given a summable function $f \in L^1(\mathbb{R})$ there are two natural ways to assign to the function a canonical set of full Lebesgue measure Λ, so that f is in some natural way defined at *every* point of Λ (see the first paragraph of [AD, p. 384]).

The first way is this. If $f \in L^1(\mathbb{R})$, then one can define a set of full Lebesgue measure $\Lambda'(f)$ as the set of all those numbers x at which the function

$$\int_0^x f(t)\,dt$$

is differentiable. Lebesgue's differentiation theorem says this set is a full one. If $x \in \Lambda'(f)$, then one can define $f(x)$ by the formula

$$f(\lambda) := \frac{d}{d\lambda}\int_0^\lambda f(x)\,dx.$$

However, there is another canonical set of full Lebesgue measure, associated with f :

$$\Lambda(f) := \left\{\lambda \in \mathbb{R} \colon \ \lim_{y\to 0^+} \operatorname{Im} \mathcal{C}_F(\lambda+iy) \text{ exists}\right\},$$

where $F(\lambda) = \int_0^\lambda f(x)\,dx$ and $\mathcal{C}_F(z)$ is the Cauchy-Stieltjes transform of F. That $\Lambda(f)$ is a full set follows from Theorem 2.3. For any $\lambda \in \Lambda(f)$ one can define the value $f(\lambda)$ of the function f at λ by the formula

$$f(\lambda) := \frac{1}{\pi}\operatorname{Im} \mathcal{C}_F(\lambda+i0) := \frac{1}{\pi}\lim_{y\to 0^+}\operatorname{Im} \mathcal{C}_F(\lambda+iy) = \lim_{y\to 0^+} f * P_y(\lambda). \tag{10}$$

Since $\frac{1}{\pi}\operatorname{Im}\mathcal{C}_F(\lambda+iy)$ is the Poisson kernel of F (see (9)), it follows from Theorem 2.3 that the two explicit summable functions defined in this way are equivalent.

So, from now on, all summable functions f are understood in this sense (if not stated otherwise): f is a function on the full set $\Lambda(f)$ defined by (10). Probably, it is worth to stress again that in this definition by a function we mean a function.

2.2.6. *De la Vallée Poussin decomposition theorem.* This is the following theorem (see e.g. [Sa, Theorem IV.9.6]):

Theorem 2.5. *Let m be a finite signed measure. Let $|m|$ be the total variation of m. Let $E_{-\infty}$ (respectively, $E_{+\infty}$) be the set where the derivative of the distribution function F_m*

of m is $-\infty$ (respectively, $+\infty$). If X is a Borel subset of $\mathbb{R}$, then

$$m(X) = m(X \cap E_{-\infty}) + m(X \cap E_{+\infty}) + \int_X F'_m(t)\,dt$$

and

$$|m|\,(X) = |m(X \cap E_{-\infty})| + m(X \cap E_{+\infty}) + \int_X |F'_m(t)|\,dt.$$

Remark. The formulation of [Sa, Theorem IV.9.6] contains an additional condition that F_m is continuous at every point of X. This condition is obviously redundant.

2.2.7. *Standard supports of measures.* If m is a finite signed measure, then its Cauchy-Stieltjes transform $\mathcal{C}_m(z)$ is defined as the Cauchy-Stieltjes transform of its distribution function

$$F_m(x) = m((-\infty, x)).$$

That is,

$$\mathcal{C}_m(z) := \int_{-\infty}^{\infty} \frac{m(dx)}{x - z}.$$

A finite signed measure has a natural decomposition

$$m = m^{(a)} + m^{(s)}$$

into the sum of an absolutely continuous measure $m^{(a)}$ and a singular measure $m^{(s)}$. The signed measures $m^{(a)}$ and $m^{(s)}$ are mutually singular. It is desirable to split the set of real numbers $\mathbb{R}$ in some natural way, such that the first set is a Borel support of the absolutely continuous part $m^{(a)}$, while the second set is a Borel support of the singular part $m^{(s)}$ of the measure m.

It is possible to do so in several ways. The choice which suits our needs in the best way is the following. To every finite signed measure m we assign the set

$$\Lambda(m) := \{\lambda \in \mathbb{R} \colon \text{ a finite limit } \operatorname{Im} \mathcal{C}_m(\lambda + i0) \in \mathbb{R} \text{ exists}\},$$

This set was introduced by Aronszajn in [Ar].

The following theorem belongs to Aronszajn [Ar].

Theorem 2.6. *Let m be a finite signed measure. The set $\Lambda(m)$ is a full set. The complement of the set $\Lambda(m)$ is a minimal Borel support of the singular part of m.*

Proof. As it was mentioned before, that the set $\Lambda(m)$ is a full set follows from Theorem 2.3.

What we need to show is that for every bounded $\Delta \subset \Lambda(m)$ the equality

$$m(\Delta) = m^{(a)}(\Delta)$$

holds. Using notation of Theorem 2.5, it follows from Fatous's theorem that the sets

$$\Delta \cap E_{-\infty} \quad \text{and} \quad \Delta \cap E_{+\infty}$$

are empty. Indeed, Fatous's theorem implies that at points λ of $E_{\pm\infty}$ the limit $\operatorname{Im}\mathcal{C}_m(\lambda+i0)$ is infinite, while at points of $\Delta \subset \Lambda(m)$ the limit $\operatorname{Im}\mathcal{C}_m(\lambda+i0)$ is finite by definition. Consequently, Theorem 2.5 completes the proof. □

The main point of this theorem is that it gives a natural splitting of the set of real numbers $\mathbb{R}$ into two parts such that the first part supports $m^{(a)}$ and the second part supports $m^{(s)}$. Actually, the support of the singular part $\mathbb{R}\setminus\Lambda(m)$ can be made smaller. Namely, the set of all points $\lambda \in \mathbb{R}$ for which $\operatorname{Im}\mathcal{C}_m(\lambda+i0)$ equals $+\infty$ or $-\infty$ is a Borel support of the singular part of m.

The function $\operatorname{Im}\mathcal{C}_m(\lambda+iy)$ cannot grow to infinity faster than C/y. If it grows as C/y, then the point λ has a non-zero measure proportional to C. The set of points where $\operatorname{Im}\mathcal{C}_m(\lambda+iy)$ grows as C/y form a Borel support of the discrete part of m. The set of points where $\operatorname{Im}\mathcal{C}_m(\lambda+iy)$ grows to infinity slower than C/y form a Borel support of the singular continuous part of m. These Borel supports were also introduced in [Ar]. Though these supports of the singular part(s) of m are more natural and finer than $\mathbb{R}\setminus\Lambda(m)$, for the purposes of this paper the last support suffices.

Also, imposing different growth conditions on $\operatorname{Im}\mathcal{C}_m(\lambda+iy)$, such as $\operatorname{Im}\mathcal{C}_m(\lambda+iy) \sim C/y^\rho$, where $\rho \in (0,1)$, one can get further finer classification of the singular continuous spectrum, see [Ro] for details.

The set $\Lambda(m)$ is not a minimal Borel support of $m^{(a)}$, but it is not difficult to indicate a natural minimal Borel support of $m^{(a)}$ (see [Ar]):

$$\mathcal{A}(m) = \{\lambda \in \Lambda(m) : \operatorname{Im}\mathcal{C}_m(\lambda+i0) \neq 0\}.$$

This follows from the fact that for a.e. $\lambda \in \Lambda(m)$

$$F'_m(\lambda) = \frac{1}{\pi}\operatorname{Im}\mathcal{C}_m(\lambda+i0)$$

and form the fact that the function $\lambda \mapsto F'_m(\lambda)$ is a density of the absolutely continuous part of m. The number $F'_m(\lambda)$ will be considered as a standard value of the density function at points of $\Lambda(m)$.

Corollary 2.7. *Let F be a function of bounded variation on $\mathbb{R}$. For any Borel subset Δ of $\Lambda(F)$ the equalities*

$$\int_\Delta dF(\lambda) = \int_\Delta F'(\lambda)\,d\lambda = \frac{1}{\pi}\int_\Delta \operatorname{Im}\mathcal{C}_F(\lambda+i0)\,d\lambda = \frac{1}{\pi}\int_\Delta \operatorname{Im}\mathcal{C}_{F^{(a)}}(\lambda+i0)\,d\lambda$$

hold.

Proof. Let m be the (signed) measure corresponding to F. Then

$$\int_\Delta dF(\lambda) = m(\Delta) = m^{(a)}(\Delta),$$

since $\mu^{(s)}(\Delta) = 0$ by Theorem 2.6. So, the first equality follows. The second equality follows from Theorem 2.3. The last equality follows from the Lebesgue theorem: $F'(\lambda) = (F^{(a)})'(\lambda)$ for a.e. $\lambda \in \mathbb{R}$ and Theorem 2.3. □

There is another canonical full set associated with a function of bounded variation, namely, the Lebesgue set of all points where F is differentiable. But the set $\Lambda(F)$ is easier to deal with, and it seems to be more natural in the context of scattering theory.

2.3. **Bounded operators.** Let $\mathcal{H}$ be a separable Hilbert space with scalar product $\langle\cdot,\cdot\rangle$, anti-linear in the first variable (all Hilbert spaces in this paper are complex and separable). Let T be a bounded operator on $\mathcal{H}$. The (uniform) norm $\|T\|$ of a bounded operator T is defined as

$$\|T\| = \sup_{f\in\mathcal{H},\,\|f\|=1} \|Tf\|.$$

A bounded operator T in $\mathcal{H}$ is non-negative, if $\langle Tf, f\rangle \geqslant 0$ for any $f \in \mathcal{H}$.

The algebra of all bounded operators in $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. Let α run through some net of indices I.

A net of operators $T_\alpha \in \mathcal{B}(\mathcal{H})$ converges to $T \in \mathcal{B}(\mathcal{H})$ in the strong operator topology, if for any $f \in \mathcal{H}$ the net of vectors $T_\alpha f$ converges to Tf. In other words, the strong operator topology is generated by seminorms $T \mapsto \|Tf\|$, where $f \in \mathcal{H}$.

A net of operators $T_\alpha \in \mathcal{B}(\mathcal{H})$ converges to $T \in \mathcal{B}(\mathcal{H})$ in the weak operator topology, if for any $f, g \in \mathcal{H}$ the net $\langle T_\alpha f, g\rangle$ converges to $\langle Tf, g\rangle$. In other words, the weak operator topology is generated by seminorms $T \mapsto |\langle Tf, g\rangle|$, where $f, g \in \mathcal{H}$.

The adjoint T^* of a bounded operator T is the unique operator which for all $f, g \in \mathcal{H}$ satisfies the equality $\langle T^*f, g\rangle = \langle f, Tg\rangle$. A bounded operator T is self-adjoint if $T^* = T$.

If T is a bounded self-adjoint operator, then for any bounded Borel function f there is a bounded self-adjoint operator $f(T)$ (the Spectral Theorem), such that the map $f \mapsto f(T)$ is a homomorphism.

The real $\mathrm{Re}(T)$ and the imaginary $\mathrm{Im}(T)$ parts of an operator $T \in \mathcal{B}(\mathcal{H})$ are defined by

$$\mathrm{Re}(T) = \frac{T+T^*}{2} \quad \text{and} \quad \mathrm{Im}(T) = \frac{T-T^*}{2i}.$$

The real and imaginary parts are self-adjoint operators.

The absolute value $|T|$ of a bounded operator T is the operator

$$|T| = \sqrt{T^*T}.$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is Fredholm, if (1) the kernel of T

$$\ker(T) := \{f \in \mathcal{H} \colon Tf = 0\}$$

if finite-dimensional, (2) the image of T

$$\mathrm{im}(T) := \{f \in \mathcal{H} \colon\ \exists g \in \mathcal{H}\ f = Tg\}$$

is closed and (3) the orthogonal complement (that is, co-kernel $\operatorname{coker}(T)$) of $\operatorname{im}(T)$ is finite-dimensional. If T is Fredholm, then the index $\operatorname{ind}(T)$ of T is the number

$$\operatorname{ind}(T) := \dim \ker(T) - \dim \operatorname{coker}(T).$$

Theorem 2.8. *(Fredholm alternative) If K is a compact operator, then $1+K$ is Fredholm and* $\operatorname{ind}(1+K) = 0$.

In particular, if K is compact and if $1+K$ has trivial kernel, then $1+K$ is invertible.

2.4. **Self-adjoint operators.** For details regarding the material of this subsection see [RS].

Let $\mathcal{H}$ be a separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, anti-linear in the first variable.

By a linear operator T on $\mathcal{H}$ one means a linear operator from some linear manifold $\mathcal{D}(T) \subset \mathcal{H}$ to $\mathcal{H}$. The set $\mathcal{D}(T)$ is called the domain of T. A linear operator T is symmetric if its domain $\mathcal{D}(T)$ is dense and if for any $f, g \in \mathcal{D}(T)$ the equality $\langle Tf, g \rangle = \langle f, Tg \rangle$ holds. A linear operator S is an extension of a linear operator T, if $\mathcal{D}(T) \subset \mathcal{D}(S)$ and $Sf = Tf$ for all $f \in \mathcal{D}(T)$. In this case one also writes $T \subset S$ (this inclusion can be considered as inclusion of sets, if one identifies an operator with its graph). A linear operator T is closed if $f_1, f_2, \ldots \in \mathcal{D}(T)$, $f_n \to f$ and $Tf_n \to g$ as $n \to \infty$ imply that $f \in \mathcal{D}(T)$ and $Tf = g$. An operator T is closable, if it has a closed extension. For every closable operator T there exists a minimal (with respect to order $\subset$) closed extension $\overline{T}$. The adjoint T^* of a densely defined operator T is a linear operator with domain

$$\mathcal{D}(T^*) := \{g \in \mathcal{H} \colon \exists h \in \mathcal{H}\ \forall f \in \mathcal{D}(T)\ \ \langle Tf, g \rangle = \langle f, h \rangle\}\,;$$

such a vector h is unique and by definition $T^* g = h$. For every densely defined closable operator T its adjoint T^* is closed. For every densely defined operator T the inclusion $\overline{T} \subset T^{**}$ holds. A symmetric operator T satisfies $\overline{T} \subset T^*$. A symmetric operator T is called self-adjoint if $T = T^*$. So, self-adjoint operator is automatically closed.

The resolvent set $\rho(H)$ of an operator H in $\mathcal{H}$ consists of all those complex numbers $z \in \mathbb{C}$, for which the operator $H - z$ has a bounded inverse with domain dense in $\mathcal{H}$. The resolvent of an operator H is the operator

$$R_z(H) = (H - z)^{-1}, \quad z \in \rho(H).$$

The spectrum $\sigma(H)$ of H is the complement of the resolvent set $\rho(H)$, i.e. $\sigma(H) = \mathbb{C} \setminus \rho(H)$.

A closed symmetric operator H is self-adjoint if and only if $\ker(H - z) = \{0\}$ for any non-real $z \in \mathbb{C}$. The spectrum of a self-adjoint operator is a subset of $\mathbb{R}$.

Let H_0 be a self-adjoint operator with domain $\mathcal{D}(H_0)$ in $\mathcal{H}$. By $E_X = E_X^{H_0}$ we denote the spectral projection of the operator H_0, corresponding to a Borel set $X \subset \mathbb{R}$ (cf. [RS]). Usually, dependence on the operator H_0 will be omitted in the notation of the spectral projection. If $X = (-\infty, \lambda)$, then we also write $E_\lambda = E_{(-\infty,\lambda)}$.

By a subspace of a Hilbert space $\mathcal{H}$ we mean a closed linear subspace of $\mathcal{H}$.

If $f, g \in \mathcal{H}$, then the spectral measure associated with f and g is the (signed) measure

$$m_{f,g}(X) = \langle E_X f, g \rangle .$$

We also write $m_f = m_{f,f}$.

A vector f is called absolutely continuous (respectively, singular) with respect to H_0, if the spectral measure $m_f(X) = \langle E_X f, f \rangle$ is absolutely continuous (respectively, singular). The set of all vectors, absolutely continuous with respect to H_0, form a (closed) subspace of $\mathcal{H}$, denoted by $\mathcal{H}^{(a)}(H_0)$. The subspace $\mathcal{H}^{(a)}(H_0)$ is called the absolutely continuous subspace (with respect to H_0). Similarly, the set of all vectors, singular with respect to H_0, form a subspace of $\mathcal{H}$, denoted by $\mathcal{H}^{(s)}(H_0)$. The subspace $\mathcal{H}^{(s)}(H_0)$ is called the singular subspace (with respect to H_0). If there is no danger of confusion, dependence on the self-adjoint operator H_0 is usually omitted.

The absolutely continuous and singular subspaces of H_0 are invariant subspaces of H_0. That is, if $f \in \mathcal{H}^{(a)}(H_0) \cap \mathcal{D}(H_0)$ then $H_0 f \in \mathcal{H}^{(a)}(H_0)$; similarly, if $f \in \mathcal{H}^{(s)}(H_0) \cap \mathcal{D}(H_0)$ then $H_0 f \in \mathcal{H}^{(s)}(H_0)$. Also, $\mathcal{H}^{(a)}(H_0)$ and $\mathcal{H}^{(s)}(H_0)$ are orthogonal, and their direct sum is the whole $\mathcal{H}$:

$$\mathcal{H}^{(a)} \perp \mathcal{H}^{(s)}$$

and

$$\mathcal{H}^{(a)} \oplus \mathcal{H}^{(s)} = \mathcal{H}.$$

The absolutely continuous (respectively, singular) spectrum $\sigma^{(a)}(H_0)$ (respectively, $\sigma^{(s)}(H_0)$) of H_0 is the spectrum of the restriction of H_0 to $\mathcal{H}^{(a)}(H_0)$ (respectively, to $\mathcal{H}^{(s)}(H_0)$).

By $P^{(a)}(H_0)$ we denote the orthogonal projection onto the absolutely continuous subspace of H_0. If $f \in \mathcal{H}$, then by $f^{(a)}$ we denote the absolutely continuous part of f with respect to H_0, i.e. $f^{(a)} = P^{(a)} f$.

The set of all densely defined closed operators on $\mathcal{H}$ will be denoted by $\mathcal{C}(\mathcal{H})$.

2.5. **Trace-class and Hilbert-Schmidt operators.**

2.5.1. *Schatten ideals.* Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. A bounded operator $T \colon \mathcal{H} \to \mathcal{K}$ is finite-dimensional, if its image $\operatorname{im}(T)$ is finite-dimensional. A bounded operator $T \colon \mathcal{H} \to \mathcal{K}$ is compact, if one of the following equivalent conditions hold: (1) T is the uniform limit of a sequence of finite-dimensional operators; (2) the closure of the image $T(B_1)$ of the unit ball $B_1 := \{f \in \mathcal{H} \colon \|f\| \leqslant 1\}$ is compact in $\mathcal{K}$.

By $\mathcal{L}_\infty(\mathcal{H}, \mathcal{K})$ we denote the set of all compact operators from a Hilbert space $\mathcal{H}$ to a possibly another Hilbert space $\mathcal{K}$. If $\mathcal{K} = \mathcal{H}$, then we write $\mathcal{L}_\infty(\mathcal{H})$. The same agreement is used in relation to other classes of operators.

The set of compact operators $\mathcal{L}_\infty(\mathcal{H})$ is an involutive norm-closed two-sided ideal of the algebra $\mathcal{B}(\mathcal{H})$.

Let T be a compact operator from $\mathcal{H}$ to $\mathcal{K}$. The absolute value of T is the self-adjoint compact operator

$$|T| := \sqrt{T^*T}.$$

Singular numbers (or s-numbers)

$$s_1(T), s_2(T), s_3(T), \ldots$$

of the operator T are eigenvalues of $|T|$, listed as a non-increasing sequence, and such that the number of appearances of each eigenvalue is equal to the multiplicity of that eigenvalue. Every compact operator $T \in \mathcal{L}_\infty(\mathcal{H}, \mathcal{K})$ can be written in the Schmidt representation:

$$T = \sum_{n=1}^{\infty} s_n(T) \langle \varphi_n, \cdot \rangle \psi_n,$$

where (φ_n) is an orthonormal basis in $\mathcal{H}$, and (ψ_n) is an orthonormal basis in $\mathcal{K}$.

Singular numbers of a compact operator T have the following property: for any $A, B \in \mathcal{B}(\mathcal{H})$

$$s_n(ATB) \leqslant \|A\| \, \|B\| \, s_n(T). \tag{11}$$

Also, $s_n(A) = s_n(A^*)$.

Let $p \in [1, \infty)$. By $\mathcal{L}_p(\mathcal{H})$ we denote the set of all compact operators T in $\mathcal{H}$, such that

$$\|T\|_p := \left(\sum_{n=1}^{\infty} s_n^p(T) \right)^{1/p} < \infty.$$

The space $(\mathcal{L}_p(\mathcal{H}), \|\cdot\|_p)$ is an *invariant operator ideal*; this means that

(1) $\mathcal{L}_p(\mathcal{H})$ is a Banach space,
(2) $\mathcal{L}_p(\mathcal{H})$ is a $*$-ideal, that is, if $T \in \mathcal{L}_p(\mathcal{H})$ and $A, B \in \mathcal{B}(\mathcal{H})$, then $T^*, AT, TA \in \mathcal{L}_p(\mathcal{H})$,
(3) for any $T \in \mathcal{L}_p(\mathcal{H})$ and $A, B \in \mathcal{B}(\mathcal{H})$ the following inequalities hold:

$$\|T\|_p \geqslant \|T\|, \ \|T^*\|_p = \|T\|_p \ \text{and} \ \|ATB\|_p \leqslant \|A\| \, \|T\|_p \, \|B\|.$$

A norm which satisfies the above three conditions is called unitarily invariant norm. The ideal $\mathcal{L}_p(\mathcal{H})$ is called the Schatten ideal of p-summable operators.

Note that for the definition of the singular numbers $s_1(T), s_2(T), \ldots$ of an operator T it is immaterial whether T acts from $\mathcal{H}$ to $\mathcal{H}$, or maybe from $\mathcal{H}$ to another Hilbert space $\mathcal{K}$. In the latter case we write $T \in \mathcal{L}_p(\mathcal{H}, \mathcal{K})$.

2.5.2. *Trace-class operators.* Operators from $\mathcal{L}_1(\mathcal{H})$ are called trace-class operators. For trace-class operators T one defines the trace $\mathrm{Tr}(T)$ by the formula

$$\mathrm{Tr}(T) = \sum_{j=1}^{\infty} \langle T\varphi_j, \varphi_j \rangle, \tag{12}$$

where $\{\varphi_j\}_{j=1}^{\infty}$ is an arbitrary orthonormal basis of $\mathcal{H}$. Sometimes we write $\mathrm{Tr}_{\mathcal{H}}(T)$ instead of $\mathrm{Tr}(T)$ to indicate the Hilbert space which T acts on. For a trace-class operator T the series above is absolutely convergent and is independent from the choice of the basis $\{\varphi_j\}_{j=1}^{\infty}$. The trace $\mathrm{Tr}\colon \mathcal{L}_1(\mathcal{H}) \to \mathbb{C}$ is a continuous linear functional, which satisfies the equality

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA),$$

whenever both products AB and BA are trace-class. In particular, the above equality holds, if A is trace-class and B is a bounded operator.

The norm $\|\cdot\|_1$ is called trace-class norm. For any trace-class operator T the following equality holds:

$$\|T\|_1 = \mathrm{Tr}(|T|).$$

More generally,

$$\|T\|_p = \left(\mathrm{Tr}(|T|^p)\right)^{1/p}.$$

The Lidskii theorem asserts that for any trace-class operator T

$$\mathrm{Tr}(T) = \sum_{j=1}^{\infty} \lambda_j,$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ is the list of eigenvalues of T counting multiplicities.[1]

The dual of the Banach space $\mathcal{L}_1(\mathcal{H})$ is the algebra of all bounded operators $\mathcal{B}(\mathcal{H})$ with uniform norm $\|\cdot\|$: every continuous linear functional on $\mathcal{L}_1(\mathcal{H})$ has the form

$$T \mapsto \mathrm{Tr}(AT)$$

for some bounded operator $A \in \mathcal{B}(\mathcal{H})$, and, vice versa, any functional of this form is continuous.

2.5.3. *Hilbert-Schmidt operators.* Operators from $\mathcal{L}_2(\mathcal{H})$ are called Hilbert-Schmidt operators. The norm

$$\|T\|_2 = \sqrt{\mathrm{Tr}(|T|^2)}$$

is also called Hilbert-Schmidt norm. For a Hilbert-Schmidt operator $T \in \mathcal{L}_2(\mathcal{H})$ and any orthonormal basis (φ_j) of $\mathcal{H}$ the following equality holds:

$$\|T\|_2^2 = \sum_{j=1}^{\infty} \|T\varphi_j\|^2. \tag{13}$$

If S, T are Hilbert-Schmidt operators, then the product ST is trace-class and the following inequality holds:

$$\|ST\|_1 \leqslant \|S\|_2 \|T\|_2.$$

[1]By multiplicity of an eigenvalue λ_j of T we always mean algebraic multiplicity; that is, the dimension of the vector space $\left\{f \in \mathcal{H}\colon \exists\, k = 1, 2, \ldots\ (T - \lambda_j)^k f = 0\right\}$

This assertion is a particular case of the more general Hölder inequality which follows. Let $p, q \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $S \in \mathcal{L}_p(\mathcal{H})$ and $T \in \mathcal{L}_q(\mathcal{H})$, then ST is trace-class and

$$\|ST\|_1 \leqslant \|S\|_p \|T\|_q ,$$

where $\|\cdot\|_\infty$ means the usual operator norm.

The ideal $\mathcal{L}_2(\mathcal{H})$ is actually a Hilbert space with scalar product

$$\langle S, T\rangle = \mathrm{Tr}(S^*T).$$

So, the dual of $\mathcal{L}_2(\mathcal{H})$ is $\mathcal{L}_2(\mathcal{H})$ itself.

2.5.4. *Fredholm determinant.* Let (φ_j) be an orthonormal basis in $\mathcal{H}$. If T is a trace-class operator, then one can define the determinant $\det(1+T)$ of $1+T$ by the formula

$$\det(1+T) = \lim_{n\to\infty} \det\big(\,\langle (1+T)\varphi_i, \varphi_j\rangle\,\big)_{i,j=1}^n,$$

where the determinant in the right hand side is the usual finite-dimensional determinant. For any trace-class operator T the limit in the right hand side exists and it does not depend on the choice of the orthonormal basis (φ_j).

The following formula holds:

$$\det(1+T) = \prod_{j=1}^{\infty} (1+\lambda_j),$$

where $\lambda_1, \lambda_2, \lambda_3, \dots$ is the list of eigenvalues of T counting multiplicities.

The determinant has the product property: for any trace-class operators S, T the equality holds:

$$\det\Big((1+S)(1+T)\Big) = \det(1+S)\det(1+T).$$

The determinant is a continuous functional (not linear, of course) on the ideal of trace-class operators $\mathcal{L}_1(\mathcal{H})$.

2.5.5. *The Birman-Koplienko-Solomyak inequality.* The following assertion is called the Birman-Koplienko-Solomyak inequality[2] (cf. [BKS]).

Theorem 2.9. *If A and B are two non-negative trace-class operators, then*

$$\left\|\sqrt{A} - \sqrt{B}\right\|_2 \leqslant \left\|\sqrt{|A-B|}\right\|_2 .$$

In [BKS] a more general inequality is proved:

$$\|A^p - B^p\|_{\mathfrak{S}} \leqslant \||A-B|^p\|_{\mathfrak{S}} ,$$

where $p \in (0,1]$ and $\|\cdot\|_{\mathfrak{S}}$ is any unitarily invariant norm.

[2] I thank Prof. P. G. Dodds for pointing out to this inequality

In [An], T. Ando (who was not aware of the paper [BKS] at the time of writing [An]) proved the following inequality

$$\|f(A) - f(B)\|_{\mathfrak{S}} \leqslant \|f(|A - B|)\|_{\mathfrak{S}},$$

where $f\colon [0,\infty) \to [0,\infty)$ is any operator-monotone function, that is, a function with property: if $A \geqslant B \geqslant 0$, then $f(A) \geqslant f(B) \geqslant 0$. Ando's inequality implies the Birman-Koplienko-Solomyak inequality, since $f(x) = x^p$ with $p \in (0,1]$ is operator-monotone. Ando's inequality was generalized to the setting of semifinite von Neumann algebras in [DD].

Lemma 2.10. *If $A_n \geqslant 0$, $A_n \in \mathcal{L}_1$ for all $n = 1, 2\ldots$, and if $A_n \to A$ in $\mathcal{L}_1$, then $\sqrt{A_n} \to \sqrt{A}$ in $\mathcal{L}_2$.*

Proof. It follows from Theorem 2.9, that

$$\left\|\sqrt{A_n} - \sqrt{A}\right\|_2 \leqslant \left\|\sqrt{|A_n - A|}\right\|_2 = \sqrt{\|A_n - A\|_1} \to 0,$$

as $n \to \infty$. The proof is complete. □

2.6. **Direct integral of Hilbert spaces.** In this subsection I follow [BS, Chapter 7].

Let Λ be a Borel subset of $\mathbb{R}$ with Borel measure ρ (we do not need more general measure spaces here), and let

$$\{\mathfrak{h}_\lambda, \ \lambda \in \Lambda\}$$

be a family of Hilbert spaces, such that the dimension function

$$\Lambda \ni \lambda \mapsto \dim \mathfrak{h}_\lambda \in \{0, 1, 2\ldots, \infty\}$$

is measurable. Let Ω_0 be a countable family of vector-functions (or sections) $f_1, f_2, \ldots$ such that to each $\lambda \in \Lambda$ f_j assigns a vector $f_j(\lambda) \in \mathfrak{h}_\lambda$.

Definition 2.11. *A family $\Omega_0 = \{f_1, f_2, \ldots\}$ of vector-functions is called a* measurability base*, if it satisfies the following two conditions:*

1. *for a.e. $\lambda \in \Lambda$ the set $\{f_j(\lambda) \colon j \in \mathbb{N}\}$ generates the Hilbert space $\mathfrak{h}_\lambda$;*
2. *the scalar product $\langle f_i(\lambda), f_j(\lambda)\rangle$ is ρ-measurable for all $i, j = 1, 2, \ldots$*

A vector-function $\Lambda \ni \lambda \mapsto f(\lambda) \in \mathfrak{h}_\lambda$ is called *measurable*, if $\langle f(\lambda), f_j(\lambda)\rangle$ is measurable for all $j = 1, 2, \ldots$. The set of all measurable vector-functions is denoted by $\hat{\Omega}_0$.

A measurability base $\{e_j(\cdot)\}$ is called *orthonormal*, if for ρ-a.e. λ the system $\{e_j(\lambda)\}$ — after throwing out zero vectors out of it — forms an orthonormal base of the fiber Hilbert space $\mathfrak{h}_\lambda$. (This definition of an orthonormal measurability base slightly differs from the one given in [BS]).

If we have a sequence $f_1, f_2, \ldots$ of vectors in a Hilbert space, then by Gram-Schmidt orthogonalization process we mean the following procedure: for $n = 1, 2, \ldots$ we replace the function f_n by zero vector if f_n is a linear combination (in particular, if $f_n = 0$) of $f_1, \ldots, f_{n-1}$, otherwise, we replace f_n by the unit vector e_n which is a linear combination

of $f_1, \ldots, f_n$, which is orthogonal to all $f_1, \ldots, f_{n-1}$ and which satisfies the inequality $\langle e_n, f_n \rangle > 0$. Obviously, the systems $\{f_j\}$ and $\{e_j\}$ generate the same linear subspace of the Hilbert space.

Lemma 2.12. [BS, Lemma 7.1.1] *If Ω_0 is a measurability base, then there exists an orthonormal measurability base Ω_1 such that $\hat{\Omega}_0 = \hat{\Omega}_1$, that is, sets of measurable vector-functions generated by Ω_0 and Ω_1 coincide.*

Proof. We apply the Gram-Schmidt orthogonalization procedure to the vectors $f_1(\lambda), f_2(\lambda), \ldots$ for a.e. $\lambda \in \Lambda$:

$$e_n(\lambda) = \sum_{j=1}^{n} a_{nj}(\lambda) f_j(\lambda).$$

The functions $a_{nj}(\lambda)$, being algebraic expressions of the scalar products $\langle f_i(\lambda), f_j(\lambda) \rangle$, are measurable. Let $\Omega_1 = \{e_j\}$. The set $\{e_j(\cdot)\}$ is an orthonormal measurability base by construction. From the last equality it follows that every Ω_0-measurable function is also Ω_1-measurable.

Now it is left to show that any Ω_1-measurable function $h(\cdot)$ is also Ω_0-measurable. If $g(\cdot)$ is an Ω_0-measurable function, then for a.e. λ

$$\langle h(\lambda), g(\lambda) \rangle = \sum_{k=1}^{\infty} \langle h(\lambda), e_k(\lambda) \rangle \langle e_k(\lambda), g(\lambda) \rangle, \tag{14}$$

in particular, the last equality holds for $g = f_j$. It follows that every Ω_1-measurable function h is also Ω_0-measurable. $\square$

Lemma 2.13. [BS, Corollary 7.1.2] *(i) If $f(\cdot)$ and $g(\cdot)$ are measurable vector-functions, then the function $\Lambda \ni \lambda \mapsto \langle f(\lambda), g(\lambda) \rangle_{\mathfrak{h}_\lambda}$ is also measurable.*
(ii) If $f(\cdot)$ is a measurable vector-function, then the function $\Lambda \ni \lambda \mapsto \|f(\lambda)\|_{\mathfrak{h}_\lambda}$ is measurable.

Proof. (i) By Lemma 2.12, one can assume that f and g are $\{e_j\}$-measurable. The claim then follows from (14). (ii) follows from (i). $\square$

Two measurable functions are *equivalent*, if they coincide for ρ-a.e. $\lambda \in \Lambda$.

The direct integral of Hilbert spaces

$$\mathcal{H} = \int_\Lambda^\oplus \mathfrak{h}_\lambda \, \rho(d\lambda) \tag{15}$$

consists of all (equivalence classes of) measurable vector-functions $f(\lambda)$, such that

$$\|f\|_{\mathcal{H}}^2 := \int_\Lambda \|f(\lambda)\|_{\mathfrak{h}_\lambda}^2 \, \rho(d\lambda) < \infty.$$

The scalar product of $f, g \in \mathcal{H}$ is given by the formula

$$\langle f, g\rangle_{\mathcal{H}} = \int_\Lambda \langle f(\lambda), g(\lambda)\rangle_{\mathfrak{h}_\lambda} \, \rho(d\lambda).$$

The set of square-summable vector-functions with this scalar product is a Hilbert space.

Lemma 2.14. [BS, Lemma 7.1.5] *Let $\{\mathfrak{h}_\lambda \colon \lambda \in \Lambda\}$ be a family of Hilbert spaces with an orthogonal measurability base $\{e_j(\cdot)\}$, let $f_0 \in L^2(\Lambda, d\rho)$ be a fixed function, such that $f_0 \neq 0$ for ρ-a.e. λ. Then the set of functions*

$$\{f_0(\lambda)\chi_\Delta(\lambda)e_j(\lambda)\colon j = 1, 2, \ldots, \Delta \text{ is a Borel subset of } \Lambda\}$$

is dense in the Hilbert space (15).

There is an example of the direct integral of Hilbert spaces relevant to this paper (cf. e.g. [BS, Chapter 7]). Let $\mathfrak{h}$ be a fixed Hilbert space, let $\{\mathfrak{h}_\lambda, \ \lambda \in \Lambda\}$ be a family of subspaces of $\mathfrak{h}$ and let P_λ be the orthogonal projection onto $\mathfrak{h}_\lambda$. Let the operator-function P_λ, $\lambda \in \Lambda$, be weakly measurable. Let (ω_j) be an orthonormal basis in $\mathfrak{h}$. The family of vector-functions $f_j(\lambda) = \{P_\lambda \omega_j\}$ is a measurability base for the family of Hilbert spaces $\{\mathfrak{h}_\lambda, \ \lambda \in \Lambda\}$. The direct integral of Hilbert spaces (15) corresponding to this family is naturally isomorphic (in an obvious way) to the subspace of $L^2(\Lambda, \mathfrak{h})$, which consists of all measurable square integrable vector-functions $f(\cdot)$, such that $f(\lambda) \in \mathfrak{h}_\lambda$ for a.e. $\lambda \in \Lambda$ [BS, Chapter 7].

One of the versions of the Spectral Theorem says that for any self-adjoint operator H in $\mathcal{H}$ there exists a direct integral of Hilbert spaces (15) and an isomorphism

$$\mathcal{F} \colon \mathcal{H} \to \mathcal{H},$$

such that H_0 is diagonalized in this representation:

$$\mathcal{F}(Hf)(\lambda) = \lambda \mathcal{F}(f)(\lambda), \ f \in \operatorname{dom}(H),$$

for ρ-a.e. $\lambda \in \Lambda$.

2.7. **Operator-valued holomorphic functions.** In this subsection I follow mainly Kato's book [Ka_2]. Proofs and details can be found in this book of Kato. See also [HPh, Chapter III].

Let X be a Banach space. Let G be a region (open connected subset) of the complex plane $\mathbb{C}$. A vector-function $f \colon G \to X$ is called *holomorphic* (or strongly holomorphic), if for every $z \in G$ the limit

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. A vector-function $f \colon G \to X$ is holomorphic is and only if it is weakly holomorphic; that is, if for any continuous linear functional l on X the function $l(f(z))$ is holomorphic in G. The proof can be found in [Ka_2, Theorem III.1.37] (see also [Ka_2, Theorem III.3.12], [RS]).

A vector-function $f\colon G \to X$ is holomorphic at $z_0 \in G$ if and only if f is analytic at z_0, that is, if f admits a power series representation

$$f(z) = f_0 + (z - z_0) f_1 + (z - z_0)^2 f_2 + \ldots + (z - z_0)^n f_n + \ldots$$

with a non-zero radius of convergence, where $f_0, f_1, \ldots \in X$.

In this paper we consider only holomorphic families of compact operators on a Hilbert space. In one occasion we consider also a holomorphic family of operators of the form $1 + T(z)$, where $T(z)$ is a holomorphic family of compact operators.

Let $T\colon G \to \mathcal{L}_\infty(\mathcal{H})$ be a holomorphic family of compact operators. Let $z \in G$ and let Γ be a piecewise smooth contour in the resolvent set $\rho(T)$ of T. Assume that there is only a finite number of eigenvalues (counting multiplicities) $\lambda_1(z), \lambda_2(z), \ldots, \lambda_h(z)$ of T inside of Γ. The operator

$$P(z) = \frac{1}{2\pi i} \int_\Gamma (\zeta - T(z))^{-1} \, d\zeta.$$

is an idempotent operator[3] (an idempotent operator is a bounded operator E which satisfies the equality $E^2 = E$), corresponding to the set of eigenvalues $\lambda_1(z), \lambda_2(z), \ldots, \lambda_h(z)$. The idempotent $P(z)$ is called the Riesz idempotent operator. By the Cauchy theorem, $P(z)$ does not change, if Γ is changed continuously inside the resolvent set of T. The range of $P(z)$ is the direct sum of root spaces of eigenvalues $\lambda_1(z), \lambda_2(z), \ldots, \lambda_h(z)$.

Let $z \in G$. If $\lambda_j(z)$ is a simple (that is, of algebraic multiplicity 1) non-zero eigenvalue of $T(z)$, then in some neighbourhood of z it depends holomorphically on z and remains to be simple. So does the idempotent operator $P_j(z)$ associated with the eigenvalue $\lambda_j(z)$. In particular, the eigenvector $v_j(z)$, corresponding to $\lambda_j(z)$, is also a holomorphic function in a neighbourhood of z.

The situation is not so simple, if the eigenvalue $\lambda_j(z)$ is not simple at some point $z_0 \in G$. In this case in a neighbourhood of z_0 the eigenvalue $\lambda_j(z)$ splits (more exactly, may split and most likely does split) into several different eigenvalues $\lambda_{z_0,1}(z), \lambda_{z_0,2}(z), \ldots, \lambda_{z_0,p}(z)$, where p is the multiplicity of $\lambda_j(z_0)$. The functions $\lambda_{z_0,1}(z), \lambda_{z_0,2}(z), \ldots, \lambda_{z_0,p}(z)$ represent branches of a multi-valued holomorphic function with branch point z_0. So, they can have an algebraic singularity at z_0, though they are still continuous at z_0. The idempotent of the whole group of eigenvalues $\lambda_{z_0,1}(z), \lambda_{z_0,2}(z), \ldots, \lambda_{z_0,p}(z)$ is holomorphic in a neighbourhood of z_0; but the idempotent of a subgroup of the group firstly is not defined at z_0 and secondly as $z \to z_0$ it (more exactly, its norm) may go to infinity — that is, it can have a pole at z_0 (see e.g. [Ka$_2$, Theorem II.1.9]). Note that this is possible since an idempotent is not necessarily self-adjoint.

All these potentially "horrible" things cannot happen, if the holomorphic family of operators $T(z)$ is *symmetric*. This means that the region G has a non-empty intersection with the real-axis $\mathbb{R}$ and for $\operatorname{Im} z = 0$ the operator $T(z)$ is self-adjoint, or — at the very least — normal. Fortunately, in this paper we shall deal only with such symmetric families of holomorphic functions. Namely, if the family $T(z)$ is symmetric, then (1) eigenvalues

[3] We do not use the word projection here, since by projection we mean an orthogonal idempotent.

$\lambda_1(z), \lambda_2(z), \lambda_3(z), \ldots$ of $T(z)$ are analytic functions for real values of z (more exactly, they can be enumerated at every point z in such a way that they become analytic) (2) the eigenvectors $v_1(z), v_2(z), v_3(z), \ldots$ of $T(z)$ corresponding to those eigenvalues are analytic as well. The eigenvectors admit analytic continuation to a point z_0, where some eigenvalue is not simple, since in this case all Riesz idempotents of the group of isolated eigenvalues are orthogonal, and — as a consequence — bounded. So, the Riesz idempotents cannot have a singularity at z_0 and thus are analytic at z_0. It follows that the eigenvalues are also analytic.

For details see Kato's book.

In the light of the preceding explanatory text, the following two lemmas should not seem surprising.

Lemma 2.15. *Let* $A\colon [0,1) \ni y \mapsto A_y \in \mathcal{L}_1(\mathcal{H}),\ A_y \geqslant 0.$

(i) *If A_y is a real-analytic function for $y > 0$ with values in $\mathcal{L}_1$, then $\sqrt{A_y}$ is a real-analytic function for $y > 0$ with values in $\mathcal{L}_2$.*
(ii) *If, moreover, A_y is continuous at $y = 0$ in $\mathcal{L}_1$, then $\sqrt{A_y}$ is continuous at $y = 0$ in $\mathcal{L}_2$.*

Proof. (i) follows from equivalence of the weak and the strong analyticity. Indeed, analyticity of A_y implies weak analyticity of A_y, i.e. that for any $X \in \mathcal{B}(\mathcal{H})$ the function $\mathrm{Tr}(XA_y)$ is analytic. Choosing X to be the projection to the eigenvector corresponding to α_j, we conclude that all eigenvalue functions $\alpha_j(\lambda)$ of A_y are analytic. Hence, the eigenvalue functions $\sqrt{\alpha_j(\lambda)}$ (probably, it is not pointless to mention, that the square root here is the arithmetic one) of $\sqrt{A_y}$ are also analytic. It follows that for any $D \in \mathcal{L}_2(\mathcal{H})$ the function $\mathrm{Tr}(D\sqrt{A_y})$ is analytic. This means that $\sqrt{A_y}$ is weakly $\mathcal{L}_2(\mathcal{H})$-analytic. Hence, $\sqrt{A_y}$ is also strongly $\mathcal{L}_2(\mathcal{H})$-analytic. (ii) follows from Lemma 2.10. $\square$

Theorem 2.16. *Let A_y, $y \in [0,1)$, be a family of non-negative Hilbert-Schmidt (respectively, compact) operators, real-analytic in $\mathcal{L}_2$ (respectively, in $\|\cdot\|$) for $y > 0$. Then there exists a family $\{e_j(y)\}$ of orthonormal bases such that all vector-functions $(0,1) \ni y \mapsto e_j(y), j = 1, 2, \ldots,$ are real-analytic functions, as well as the corresponding eigenvalue functions $\alpha_j(y)$. Moreover, if A_y is continuous at $y = 0$ in the Hilbert-Schmidt norm, then all eigenvalue functions $\alpha_j(y)$ are also continuous at $y = 0$, and if $\alpha_j(0) > 0$, then the corresponding eigenvector function $e_j(y)$ can also be chosen to be continuous at $y = 0$.*

Proof. The first part follows from [Ka$_2$, Theorem II.6.1] and [Ka$_2$, §II.6.2], cf. also [Ka$_2$, §VII.3]. Continuity of $\alpha_j(y)$ at $y = 0$ follows from upper semi-continuity of the spectrum [Ka$_2$, §IV.3.1]. That the eigenvalue functions $e_j(y)$ can be chosen to be continuous at $y = 0$, provided that $\alpha_j(0) > 0$, follows from [Ka$_2$, §VII.3] (cf. also [Ka$_2$, Theorem IV.3.16]). $\square$

2.7.1. *Operator-valued meromorphic functions.* Let G be a region in $\mathbb{C}$. Let $z_0 \in G$ and let $T\colon G \setminus \{z_0\} \to \mathcal{B}(\mathcal{H})$ be a holomorphic family of bounded operators in a deleted neighbourhood of z_0. Then T admits a Laurent expansion:

$$T(z) = \sum_{n=-\infty}^{\infty} (z-z_0)^n T_n,$$

where T_n are bounded operators.

Proof. Let $f, g \in \mathcal{H}$. Then the scalar function $\langle T(z)f, g\rangle$ is holomorphic in a deleted neighbourhood of z_0 and so it admits a Laurent expansion

$$\langle T(z)f, g\rangle = \sum_{n=-\infty}^{\infty} a_n(f,g)(z-z_0)^n \tag{16}$$

in the deleted neighbourhood. The functional a_n is obviously bilinear. It is also bounded. To see this we multiply both sides of the last formula by $(z-z_0)^{-n-1}$ and integrate it over a circle $\Gamma = \{z \in G\colon\ |z-z_0| = r\}$ with small enough r (so that Γ and its interior $\subset G$) to get

$$a_n(f,g) = \frac{1}{2\pi i}\int_\Gamma (z-z_0)^{-n-1}\left\langle T(z)f, g\right\rangle\, dz.$$

Since $T(z)$ is continuous on Γ, the maximum M of its norm $\|T(z)\|$ on Γ is finite. It follows that $\|a_n\|$ is bounded by $\frac{M}{r^n}$.

So, each a_n can be identified with a bounded operator T_n; that is, for any $f, g \in \mathcal{H}$ the equality $\langle T_n f, g\rangle = a_n(f,g)$ holds. Since $\langle T(z)f, g\rangle$ is uniformly bounded on any compact subset of $G \setminus \{z_0\}$, it follows from (16) and the Banach-Steinhaus principle that the operator Laurent series

$$\sum_{n=-\infty}^{\infty} (z-z_0)^n T_n$$

converges at all points close enough to z_0 and its sum is equal to $T(z)$. The proof is complete.

Let $N = \min\{n\colon T_n \neq 0\}$. If $N > -\infty$, then $T(z)$ is said to have a pole of order N at z_0.

2.7.2. *Analytic Fredholm alternative.* This is the following theorem (see e.g. [RS, Theorem VI.14], [Y, Theorem 1.8.2]).

Theorem 2.17. *Let G be an open connected subset of $\mathbb{C}$. Let $T\colon G \to \mathcal{L}_\infty(\mathcal{H})$ be a holomorphic family of compact operators in G. If the family of operators $1 + T(z)$ is invertible at some point $z_1 \in G$, then it is invertible at all points of G except the discrete set*

$$\mathcal{N} := \{z \in G\colon 1 \in \sigma(T(z))\}.$$

Further, the operator-function $F(z) := (1+T(z))^{-1}$ is meromorphic and the set of its poles is $\mathcal{N}$. Moreover, in the expansion of $F(z)$ in a Laurent series in a neighbourhood

of any point $z_0 \in \mathcal{N}$ the coefficients of negative powers of $z - z_0$ are finite dimensional operators.

2.8. **The limiting absorption principle.** We recall two theorems from [Y] (cf. also [BW]), which are absolutely crucial for this paper. They were established by L. de Branges [B] and M. Sh. Birman and S. B. Èntina [BE].

Theorem 2.18. [Y, Theorem 6.1.5] *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. Suppose H_0 is a self-adjoint operator in the Hilbert space $\mathcal{H}$ and $F\colon \mathcal{H} \to \mathcal{K}$ is a Hilbert-Schmidt operator. Then for a.e. $\lambda \in \mathbb{R}$ the operator-valued function $FE_\lambda^{H_0}F^* \in \mathcal{L}_1(\mathcal{K})$ is differentiable in the trace-class norm, the operator-valued function $F \operatorname{Im} R_{\lambda+iy}(H_0)F^*$ has a limit in the trace-class norm as $y \to 0$, and*

$$\frac{1}{\pi} \lim_{y\to 0} F \operatorname{Im} R_{\lambda+iy}(H_0)F^* = \frac{d}{d\lambda}(FE_\lambda F^*), \tag{17}$$

where the limit and the derivative are taken in the trace-class norm.

Theorem 2.19. [Y, Theorem 6.1.9] *Suppose H_0 is a self-adjoint operator in a Hilbert space $\mathcal{H}$ and $F \in \mathcal{L}_2(\mathcal{H}, \mathcal{K})$. Then for a.e. $\lambda \in \mathbb{R}$ the operator-valued function $FR_{\lambda\pm iy}(H_0)F^*$ has a limit in $\mathcal{L}_2(\mathcal{K})$ as $y \to 0$.*

S. N. Naboko has shown that in this theorem the convergence in $\mathcal{L}_2(\mathcal{K})$ can be replaced by the convergence in $\mathcal{L}_p(\mathcal{K})$ with any $p > 1$. In general, the convergence in $\mathcal{L}_1(\mathcal{K})$ does not hold (cf. [N, N_2, N_3]).

3. Framed Hilbert space

3.1. **Definition.** In this section we introduce the so called framed Hilbert space and study several objects associated with it. Before giving formal definition, I would like to explain the idea which led to this notion.

Let H_0 be a self-adjoint operator on a Hilbert space $\mathcal{H}$, and let H_1 be its trace-class perturbation. Our ultimate purpose is to explicitly define the wave matrix $w_\pm(\lambda; H_1, H_0)$ at a fixed point λ of the spectral line. The wave matrix $w_\pm(\lambda; H_1, H_0)$ acts between fiber Hilbert spaces $\mathfrak{h}_\lambda(H_0)$ and $\mathfrak{h}_\lambda(H_1)$ from the direct integrals of Hilbert spaces

$$\int_{\hat\sigma(H_0)}^{\oplus} \mathfrak{h}_\lambda(H_0)\,d\lambda \quad \text{and} \quad \int_{\hat\sigma(H_1)}^{\oplus} \mathfrak{h}_\lambda(H_1)\,d\lambda,$$

diagonalizing absolutely continuous parts of the operators H_0 and H_1, where $\hat\sigma(H_j)$ is a core of the spectrum of H_j. Before defining $w_\pm(\lambda; H_1, H_0)$, one should first define explicitly the fiber Hilbert spaces $\mathfrak{h}_\lambda(H_0)$ and $\mathfrak{h}_\lambda(H_1)$. Moreover, given a vector $f \in \mathcal{H}$, it is necessary to be able to assign an explicit value $f(\lambda) \in \mathfrak{h}_\lambda$ of the vector f at a single point $\lambda \in \mathbb{R}$. Obviously, the vectors $f(\lambda)$ generate the fiber Hilbert space $\mathfrak{h}_\lambda$. So, one of the first important questions to ask is:

$$\text{What is} \quad f(\lambda)? \tag{18}$$

Actually, since the measure $d\lambda$ in the direct integral decomposition of the Hilbert space can be replaced by any other measure $\rho(d\lambda)$ with the same spectral type, it is not difficult to see, that $f(\lambda)$ does not make sense, as it is. Indeed, let us consider an operator of multiplication by a continuous function $f(x)$ on the Hilbert space $L^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle in the complex plane. The Hilbert space $L^2(\mathbb{T})$ can be represented as a direct integral of one-dimensional Hilbert spaces $\mathfrak{h}_\lambda \simeq \mathbb{C}$:

$$L^2(\mathbb{T}) = \int_{\mathbb{T}}^{\oplus} \mathbb{C}\, d\theta.$$

(As a measurability base one can take here the system which consists of only one function, say, $e^{in\theta}$, where n is any integer; in particular, a non-zero constant function will do). Since $f(x)$ is continuous we can certainly say what is, say, $f(1)$. But the measure $d\theta$ can be replaced by any other measure of the same spectral type; for example, by

$$d\rho(\theta) = \left(2 + \sin\frac{1}{\theta}\right) d\theta.$$

The Spectral Theorem says, that the operator of multiplication M_f by $f(\theta)$ does not notice this change of measure; that is, the operator M_f will stay in the same unitary equivalence class. At the same time, now it is difficult to say what $f(1)$ is. That is, the value $f(\lambda) \in \mathfrak{h}_\lambda$ of a vector f at a point λ of the spectral line is affected by the choice of a measure in its spectral type. As a consequence, the expression $f(\lambda)$ does not make sense. The measure ρ defined by the above formula is far from being the worst scenario: instead of $\sin\frac{1}{\theta}$ one can take, say, any L^∞-function bounded below by -1. In this case, we have a difficulty to define the value of f at any point.

In order to give meaning to $f(\lambda)$, one needs to introduce some additional structure. (One can see that fixing a measure $d\rho$ in the spectral type does not help). There are different approaches to this problem. Firstly, if we try to single out what enables to give meaning to $f(\theta)$ for all θ in the case of the measure $d\theta$, we see that this additional structure is of geometric character: it is the (Riemannian) metric. The problem is that in the setting of arbitrary self-adjoint operators we don't have a metric. But the metric is fully encoded in the Dirac operator $\frac{1}{i}\frac{d}{d\theta}$ (see [C, Chapter VI]). The operator $\frac{1}{i}\frac{d}{d\theta}$ on $L^2(\mathbb{T})$ has discrete spectrum and so it is identified by a sequence of its eigenvalues and by the orthonormal basis of its eigenvectors. This type of data consisting of numbers and vectors of the Hilbert space can be easily dealt with in the abstract situation.

So, to see in another way what kind of additional structure can allow to define $f(\lambda)$, let us assume, to begin with, that there is a fixed unit vector $\varphi_1 \in \mathcal{H}$. In this case, it is possible to define the number

$$\langle f(\lambda), \varphi_1(\lambda)\rangle$$

for a.e. λ, by formula (10), since the above scalar product is a summable function of λ. Note, that neither $f(\lambda)$, nor $\varphi_1(\lambda)$ are yet defined, but their scalar product is defined.

If there are many enough (unit) vectors $\varphi_1, \varphi_2, \ldots$, then one can hope that the knowledge of all the scalar products $\langle f(\lambda), \varphi_j(\lambda)\rangle$ will allow to recover the vector $f(\lambda) \in \mathfrak{h}_\lambda$.

(Note, that we don't know yet what exactly $\mathfrak{h}_\lambda$ is). But this is still not the case. Note that the scalar product $\langle \varphi_j(\lambda), \varphi_k(\lambda)\rangle$ should satisfy the formal equality

$$\langle \varphi_j(\lambda), \varphi_k(\lambda)\rangle = \langle \varphi_j|\delta(H_0-\lambda)|\varphi_k\rangle = \frac{1}{\pi}\langle \varphi_j|\operatorname{Im}(H_0-\lambda-i0)^{-1}|\varphi_k\rangle, \tag{19}$$

where $\langle \varphi|A|\psi\rangle$ is physicists' (Dirac's) notation for $\langle \varphi, A\psi\rangle$. That this equality must hold for the absolutely continuous part $H_0^{(a)}$ can be seen from

$$\left\langle j_\varepsilon(H_0^{(a)}-\lambda)\varphi_j, \varphi_k\right\rangle = \int_{\mathbb{R}} j_\varepsilon(\mu-\lambda)\left\langle \varphi_j(\mu), \varphi_k(\mu)\right\rangle\, d\mu,$$

where j_ε is an approximate unit for the Dirac δ-function. In order to satisfy this key equality, we use an artificial trick. We assign to each vector φ_j a weight $\kappa_j > 0$ such that $(\kappa_j) = (\kappa_1, \kappa_2, \ldots) \in \ell_2$. Now, we form the matrix

$$\varphi(\lambda) := \left(\kappa_j\kappa_k\frac{1}{\pi}\langle \varphi_j|\operatorname{Im}(H_0-\lambda-i0)^{-1}|\varphi_k\rangle\right).$$

Using the limiting absorption principle (Theorem 2.18), it can be easily shown that this matrix is a non-negative trace-class matrix. Now, if we define $\varphi_j(\lambda)$ as the jth column of the square root of the matrix $\varphi(\lambda)$ over κ_j, then $\varphi_j(\lambda)$ will become an element of ℓ_2 and the equality (19) will be satisfied. For all λ from some explicit set of full Lebesgue measure, which depends only on H_0 and the data (φ_j, κ_j), this allows to define the value $f(\lambda)$ at λ for each $f = \varphi_j,\ j = 1, 2, \ldots$ and, consequently, for any vector from the dense manifold of finite linear combinations of φ_j. Finally, the fiber Hilbert space $\mathfrak{h}_\lambda$ can be defined as a linear subspace of ℓ_2 generated by $\varphi_j(\lambda)$'s.

Evidently, the data (φ_j, κ_j) can be encoded in a single Hilbert-Schmidt operator $F = \sum_{j=1}^\infty \kappa_j\langle \varphi_j, \cdot\rangle\,\psi_j$, where (ψ_j) is an arbitrary orthonormal system in a possibly another Hilbert space. Actually, in the case of $\mathcal{H} = L^2(M)$ discussed above, where M is a Riemannian manifold, F can be chosen to be the appropriate negative power of the Dirac operator D (shifted by a small scalar operator, if necessary, to make it invertible).

This justifies introduction of the following

Definition 3.1. *A* frame *in a Hilbert space $\mathcal{H}$ is a Hilbert-Schmidt operator $F\colon \mathcal{H} \to \mathcal{K}$, with trivial kernel and co-kernel, of the following form*

$$F = \sum_{j=1}^\infty \kappa_j\left\langle \varphi_j, \cdot\right\rangle \psi_j, \tag{20}$$

where $\mathcal{K}$ is another Hilbert space, and where $(\kappa_j) \in \ell_2$ is a fixed decreasing sequence of s-numbers of F, all of which are non-zero, (φ_j) is a fixed orthonormal basis in $\mathcal{H}$, and (ψ_j) is an orthonormal basis in $\mathcal{K}$.

A framed Hilbert space *is a pair $(\mathcal{H}, F)$, consisting of a Hilbert space $\mathcal{H}$ and a frame F in $\mathcal{H}$.*

Throughout this paper we shall work with only one frame F, with some restrictions imposed later on it, and κ_j, φ_j and ψ_j will be as in the formula (20).

What is important in the definition of a frame is the orthonormal basis (φ_j) and the ℓ_2-sequence of weights (κ_j) of the basis vectors. The Hilbert space $\mathcal{K}$ is of little importance, if any. For the most part of this paper, one can take $\mathcal{K} = \mathcal{H}$ and F to be self-adjoint, but later we shall see that the more general definition given above is more useful.

A frame introduces rigidity into the Hilbert space. In particular, a frame fixes a measure on the spectrum of a self-adjoint operator by the formula $\mu(\Delta) = \operatorname{Tr}(FE^H_\Delta F^*)$. In other words, a frame fixes a measure in its spectral type.

For further use, we note trivial relations

$$F\varphi_j = \kappa_j\psi_j, \quad F^*\psi_j = \kappa_j\varphi_j. \tag{21}$$

3.2. **Spectral triple associated with an operator on a framed Hilbert space.** In previous version of this paper I wrongly claimed that a framed Hilbert space was a new notion. It turns out that the notion of a frame in a Hilbert space coincides, at least formally, with the notion of a Hilbert-Schmidt rigging [BSh, Supplement 1, Definition 2.3].

Still, what is new is a geometric view-point on the notion of a framed Hilbert space (= a Hilbert space with a Hilbert-Schmidt rigging), and the purpose it is introduced for. The point is that given a self-adjoint operator H_0 on a framed Hilbert space $(\mathcal{H}, F)$, all the notions associated with it, such as the full set $\Lambda(H_0; F)$, the matrices $\varphi(\lambda)$, $\eta(\lambda)$, vectors $\varphi_j(\lambda)$, $\eta_j(\lambda)$, $b_j(\lambda)$, the fiber Hilbert space $\mathfrak{h}_\lambda$, the measurability base $\varphi_j(\cdot)$ of the direct integral $\mathcal{H}$ (43) and the direct integral itself, the operator $\mathcal{E}$, are all constructed in a constructive way. This allows to give an explicit meaning to the expression $f(\lambda)$, where f is a regular ($\in \mathcal{H}_1(F)$) vector of the Hilbert space $\mathcal{H}$, and $\lambda \in \Lambda(H_0; F)$. This becomes possible since the pair (Hilbert space, self-adjoint operator) is endowed with a geometric structure given by frame.

For example, the rigged Hilbert space $(\mathcal{H}, X)$, where $\mathcal{H} = L^2(M)$, $X = C(M)$ and M is a compact Riemannian manifold, enables to recover the topology of M [C, Chapter VI]. Choosing $X = C^r(M)$, $r = 1, 2, \ldots$, enables to recover the differentiable structure of M. But in order to recover the metric structure of M, one needs additionally a Dirac operator D [C, Chapter VI]. A frame F can be looked at as an operator, which introduces such an abstract metric structure into the pair $(H_0, \mathcal{H})$. The involutive algebra $\mathcal{A}$ of a spectral triple $(\mathcal{A}, \mathcal{H}, |F|^{-1})$ is recovered via

$$\mathcal{A} = \left\{\varphi(H) \colon \varphi \in C_b(\mathbb{R}),\ [|F|^{-1}, \varphi(H)] \in \mathcal{B}(\mathcal{H})\right\}.$$

Here the class C_b of all continuous bounded functions on $\mathbb{R}$ can be replaced by L^∞. Let us check that $\mathcal{A}$ is an algebra. If $\varphi_1(H)$, $\varphi_2(H) \in \mathcal{A}$ and α_1, $\alpha_2 \in \mathbb{C}$, then obviously $\varphi_1(H)^* = \bar{\varphi}(H) \in \mathcal{A}$ and $\alpha_1\varphi_1(H) + \alpha_2\varphi_2(H) \in \mathcal{A}$. Now, if $\varphi_1(H), \varphi_2(H) \in \mathcal{A}$, then the operator

$$[|F|^{-1}, \varphi_1(H)\varphi_2(H)] = [|F|^{-1}, \varphi_1(H)]\varphi_2(H) + \varphi_1(H)[|F|^{-1}, \varphi_2(H)]$$

is also bounded, so that $\varphi_1(H)\varphi_2(H) \in \mathcal{A}$. Consequently, $\mathcal{A}$ is an involutive algebra. The second axiom of the spectral triple is satisfied obviously, that is the resolvent $(|F|^{-1} - z)^{-1}$ of the operator $|F|^{-1}$ is compact for non-real z.

The frame vectors $\varphi_1, \varphi_2, \ldots$ are considered to be smooth or regular vectors of the Hilbert space. The weight κ_j of a frame vector φ_j indicates in a way the "measure" of smoothness. By the measure of smoothness we don't mean the degree of smoothness: all frame vectors are absolutely smooth. For instance, a sequence of vectors $e^{in\theta}$ in $L^2(\mathbb{T})$ consists of C^∞-functions and all these vectors are equally smooth. At the same time, the function $e^{i\theta}$ is obviously more "smooth" in some sense than, say, $e^{i10^{10}\theta}$. Now, if we assign to each vector $\varphi_n = e^{in\theta}$ a weight, say, $\kappa_n = \frac{1}{n}$, (or $\kappa_n = 1$ if $n = 0$) then we get a frame, and the weight of a frame vector shows how more "smooth" is the function $e^{in\theta}$ than the function $e^{i10^{10}\theta}$.

Further, according to elliptic regularity (see e.g. [RS_2, Chapter IX]), eigenvectors of the Laplace operator Δ on a compact Riemannian manifold are infinitely differentiable functions. So, if we consider an appropriate negative power of Δ (so that it becomes Hilbert-Schmidt) as a frame operator, then its eigenvectors are automatically smooth in the usual sense as well.

Finally, let us consider another example. Let $\mathcal{H} = L^2[0,1]$ and let $\varphi_0 = 1$ and let φ_n be the n-th Rademacher function; that is

$$\varphi_n(x) = \begin{cases} 1, & \text{if } n\text{-th digit in the binary representation of } x \text{ is } 1 \\ -1, & \text{if } n\text{-th digit in the binary representation of } x \text{ is } 0 \end{cases}$$

Let φ_n be assigned a weight $\frac{1}{n}$. Now, in thus defined framed Hilbert space all Rademacher functions are smooth, by definition. One way to look at why this may happen is to consider the Hilbert space $L^2([0,1] \setminus X)$ instead of $L^2[0,1]$, where X are numbers from $[0,1]$ with finite binary representation. Of course, it is possible to choose a sequence of brownian paths, apply to it the Gram-Schmidt orthogonalization process, assign to thus obtained vectors $\varphi_1, \varphi_2, \ldots$ some weights $\kappa_1, \kappa_2, \ldots$ and get in this way a frame. Now, in this frame those nowhere differentiable functions will be by definition smooth. Of course, these functions are smooth because they are defined on some very singular space (though it is rather the other way). This is in accordance with a general idea of A. Connes [C], that a spectral triple is the most general way to define geometric objects including very weird ones.

Remark. The main point of the notion of frame is to give an answer to the question (18), and, as a consequence, ultimately to be able to define the wave matrix $w_\pm(\lambda; H_1, H_0)$ for all λ from a predefined set of full Lebesgue measure. As such this notion is new, to the best knowledge of the author. There is another additional structure in the Hilbert space, a Hilbert-Schmidt rigging, which is also given by a Hilbert-Schmidt operator. At the time of writing the first version of this paper, the author was unaware of this notion (Hilbert-Schmidt rigging). Since the main aim of a rigging is to accommodate some vectors which do not fit into the Hilbert space, a frame and a rigging are obviously different notions, despite of being defined by the same type of data.

We consider self-adjoint operators in a framed Hilbert space. To the pair (self-adjoint operator, framed Hilbert space) there can be associated a lot of structures. We proceed to analysis of these structures.

3.3. **The set $\Lambda(H_0; F)$ and the matrix $\varphi(\lambda)$.** Let H_0 be a self-adjoint operator in a framed Hilbert space $(\mathcal{H}, F)$.

By $E_\lambda = E_\lambda^{H_0}$, $\lambda \in \mathbb{R}$, we denote the family of spectral projections of H_0. For any (ordered) pair of indices (i,j) one can consider a finite (signed) measure

$$m_{ij}(\Delta) := \langle \varphi_i, E_\Delta^{H_0} \varphi_j \rangle. \tag{22}$$

We denote by

$$\Lambda_0(H_0, F)$$

the intersection of all the sets $\Lambda(m_{ij})$, $i,j \in \mathbb{N}$ (see subsection 2.2.7), even though it depends only on H_0 and the vectors $\varphi_1, \varphi_2, \varphi_3, \ldots$ So, for any $\lambda \in \Lambda_0(H_0, F)$ the limit

$$\varphi_{ij}(\lambda) := \frac{1}{\pi} \kappa_i \kappa_j \langle \varphi_i, \operatorname{Im} R_{\lambda+i0}(H_0) \varphi_j \rangle$$

exists. It follows that, for any $\lambda \in \Lambda_0(H_0, F)$, one can form an infinite matrix

$$\varphi(\lambda) = (\varphi_{ij}(\lambda))_{i,j=1}^\infty .$$

Our aim is to consider $\varphi(\lambda)$ as an operator on ℓ_2. Evidently, the matrix $\varphi(\lambda)$ is symmetric in the sense that for any $i, j = 1, 2, \ldots$

$$\overline{\varphi_{ij}(\lambda)} = \varphi_{ji}(\lambda).$$

But it may turn out that $\varphi(\lambda)$ is not a matrix of a bounded, or even of an unbounded, operator on ℓ_2. So, we have to investigate the set of points, where $\varphi(\lambda)$ determines a bounded self-adjoint operator on ℓ_2. As is shown below, it turns out that $\varphi(\lambda)$ is a trace-class operator on a set of full measure.

In the following definition one of the central notions of this paper is introduced.

Definition 3.2. *The standard set of full Lebesgue measure $\Lambda(H_0; F)$, associated with a self-adjoint operator H_0 acting on a framed Hilbert space $(\mathcal{H}, F)$, consists of those points $\lambda \in \mathbb{R}$, at which the limit of $F R_{\lambda+iy}(H_0) F^*$ (as $y \to 0^+$) exists in $\mathcal{L}_2$-norm and the limit of $F \operatorname{Im} R_{\lambda+iy}(H_0) F^*$ exists in $\mathcal{L}_1$-norm.*

In other words, a number λ belongs to $\Lambda(H_0; F)$ if and only if it belongs to both sets of full measure from Theorems 2.18 and 2.19.

Proposition 3.3. *For any self-adjoint operator H_0 on a framed Hilbert space $(\mathcal{H}, F)$ the set $\Lambda(H_0; F)$ has full Lebesgue measure.*

Proof. This follows from Theorems 2.18 and 2.19. □

The following proposition gives one of the two main properties of the set $\Lambda(H_0; F)$.

Proposition 3.4. *Let H_0 be a self-adjoint operator acting on a framed Hilbert space $(\mathcal{H}, F)$. If $\lambda \in \Lambda(H_0; F)$, then the matrix $\varphi(\lambda)$ exists, is non-negative and is trace-class.*

Proof. Let $\lambda \in \Lambda(H_0; F)$. Since for $\lambda \in \Lambda(H_0; F)$ the limit

$$FR_{\lambda \pm i0}(H_0)F^* = \lim_{y \to 0^+} FR_{\lambda \pm iy}(H_0)F^*$$

exists in the Hilbert-Schmidt norm, it follows that for any pair (i, j) the limit

$$P_i^* FR_{\lambda \pm i0}(H_0)F^* P_j = \lim_{y \to 0^+} P_i^* FR_{\lambda \pm iy}(H_0)F^* P_j$$

also exists in the Hilbert-Schmidt norm, where $P_j = \langle \varphi_j, \cdot \rangle \, \psi_j$. This is equivalent to the existence of the limit

$$\langle \varphi_i, R_{\lambda \pm i0}(H_0)\varphi_j \rangle = \lim_{y \to 0^+} \langle \varphi_i, R_{\lambda \pm iy}(H_0)\varphi_j \rangle.$$

Hence, $\Lambda(H_0; F) \subset \Lambda_0(H_0, F)$; so $\varphi(\lambda)$ exists for any $\lambda \in \Lambda(H_0; F)$.

Further, existence of the limit $FR_{\lambda \pm i0}(H_0)F^*$ in the Hilbert-Schmidt norm implies that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\kappa_i \kappa_j (\varphi_i, R_{\lambda \pm i0}(H_0)\varphi_j)|^2 < \infty,$$

since the expression under the sum is the kernel of the Hilbert-Schmidt operator $FR_{\lambda \pm i0}(H_0)F^*$ in the basis (ψ_j). It is not difficult to see that this estimate implies that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\kappa_i \kappa_j (\varphi_i, \operatorname{Im} R_{\lambda + i0}(H_0)\varphi_j)|^2 < \infty.$$

Hence, $\varphi(\lambda)$ is a non-negative Hilbert-Schmidt operator on ℓ_2.

Further, since $\mathcal{L}_1(\mathcal{K})$-$F \operatorname{Im} R_{\lambda + i0}(H_0)F^*$ exists, it follows that

$$\frac{1}{\pi} \operatorname{Tr}(F \operatorname{Im} R_{\lambda + i0}(H_0)\, F^*)$$

exists (is finite). Evaluating this trace in the basis (ψ_j) we see that it is equal to the trace of $\varphi(\lambda)$. Hence, the matrix $\varphi(\lambda)$ is trace-class. $\square$

Lemma 3.5. *The operator function $\Lambda(H_0; F) \ni \lambda \mapsto \varphi(\lambda) \in \mathcal{L}_1(\ell_2)$ is measurable.*

Indeed, $\varphi(\lambda)$ is a point-wise limit of matrices with continuous matrix elements $\varphi(\lambda + iy)$.

3.4. **A core of the singular spectrum $\mathbb{R} \setminus \Lambda(H_0, F)$.** We call a null set $X \subset \mathbb{R}$ a *core* of the singular spectrum of H_0, if the operator $E^{H_0}_{\mathbb{R} \setminus X} H_0$ is absolutely continuous. Evidently, any core of the singular spectrum contains the pure point spectrum. Apart of it, a core of the singular spectrum contains a null Borel support of the singular continuous spectrum.

Lemma 3.6. *Let H_0 be a self-adjoint operator on $\mathcal{H}$ and let Λ be a full set. If $\mathbb{R} \setminus \Lambda$ is not a core of the singular spectrum of H_0, then there exists a null set $X \subset \Lambda$, such that $E_X \neq 0$.*

Proof. Let Z_a be a full set such that E_{Z_a} is the projection onto the absolutely continuous subspace of $H_0 E_\Lambda$. Such a set exists by [Y, Lemma 1.3.6]. If $\mathbb{R} \setminus \Lambda$ is not a core of the singular spectrum, then the operator $H_0 E_\Lambda$ is not absolutely continuous. So, the set $X := \Lambda \setminus Z_a$ is a null set and $E_X \neq 0$. □

Proposition 3.7. *For any self-adjoint operator H_0 on a framed Hilbert space $(\mathcal{H}, F)$, the set $\mathbb{R} \setminus \Lambda_0(H_0, F)$ is a core of the singular spectrum of H_0.*

Proof. Assume the contrary. Then by Lemma 3.6 there exists a null subset X of $\Lambda_0(H_0, F)$ such that $E_X \neq 0$. Since (φ_j) is a basis, there exists φ_j, such that $E_X \varphi_j \neq 0$. Hence, $\langle E_X \varphi_j, \varphi_j \rangle \neq 0$, that is,

$$m_{jj}^{(s)}(X) = m_{jj}(X) \neq 0,$$

where m_{jj} is the spectral measure of φ_j (see (22)). Since $X \subset \Lambda(m_{jj})$, this contradicts the fact that the complement of $\Lambda(m_{jj})$ is a Boreal support of $m_{jj}^{(s)}$ (see Theorem 2.6). □

Since $\Lambda(H_0; F) \subset \Lambda_0(H_0, F)$, it follows that

Corollary 3.8. *For any self-adjoint operator H_0 on a framed Hilbert space $(\mathcal{H}, F)$, the set $\mathbb{R} \setminus \Lambda(H_0; F)$ is a core of the singular spectrum of H_0.*

Since $\Lambda(H_0; F)$ has full measure, this corollary means that the set $\Lambda(H_0; F)$ cuts out the singular spectrum of H_0 from $\mathbb{R}$. Given a frame operator $F \in \mathcal{L}_2(\mathcal{H}, \mathcal{K})$, we consider the set $\mathbb{R} \setminus \Lambda(H_0; F)$ as a standard core of the singular spectrum of H_0, associated with the given frame F.

3.5. **The Hilbert spaces $\mathcal{H}_\alpha(F)$.** Let $\alpha \in \mathbb{R}$. In analogy with Sobolev spaces W^2_α (see e.g. [RS$_2$, §IX.6], [C$_2$]), given a framed Hilbert space $(\mathcal{H}, F)$, we introduce the Hilbert spaces $\mathcal{H}_\alpha(F)$. By definition, $\mathcal{H}_\alpha(F)$ is the completion of the linear manifold

$$\mathcal{D} = \mathcal{D}(F) := \left\{ f \in \mathcal{H} \colon \ f = \sum_{j=1}^N \alpha_j \varphi_j, \ N < \infty \right\} \tag{23}$$

in the norm

$$\|f\|_{\mathcal{H}_\alpha(F)} = \left\| |F|^{-\alpha} f \right\|,$$

with the scalar product

$$\langle f, g \rangle_{\mathcal{H}_\alpha(F)} = \left\langle |F|^{-\alpha} f, |F|^{-\alpha} g \right\rangle.$$

That is, if $f = \sum\limits_{j=1}^N \alpha_j \varphi_j$, then

$$\|f\|_{\mathcal{H}_\alpha(F)} = \left(\sum_{j=1}^N |\alpha_j|^2 \kappa_j^{-2\alpha} \right)^{1/2}. \tag{24}$$

Since F has trivial kernel, $\|\cdot\|_{\mathcal{H}_\alpha(F)}$ is indeed a norm. The scalar product of vectors $f = \sum_{j=1}^{N} \alpha_j \varphi_j$ and $g = \sum_{j=1}^{N} \beta_j \varphi_j$ in $\mathcal{H}_\alpha(F)$ is given by the formula

$$\langle f, g \rangle_{\mathcal{H}_\alpha(F)} = \sum_{j=1}^{N} \bar{\alpha}_j \beta_j \kappa_j^{-2\alpha}.$$

The Hilbert space $\mathcal{H}_\alpha(F)$ has a natural orthonormal basis $(\kappa_j^\alpha \varphi_j)$. Since

$$|F|^\gamma (\kappa_j^\alpha \varphi_j) = \kappa_j^{\alpha+\gamma} \varphi_j,$$

it follows that

Lemma 3.9. *For any $\alpha, \gamma \in \mathbb{R}$ the operator $|F|^\gamma : \mathcal{D} \to \mathcal{D}$ is unitary as an operator from $\mathcal{H}_\alpha(F)$ to $\mathcal{H}_{\alpha+\gamma}(F)$.*

It follows that all Hilbert spaces $\mathcal{H}_\alpha(F)$ are naturally isomorphic, the natural isomorphism being the appropriate power of $|F|$.

Plainly, $\mathcal{H}_0(F) = \mathcal{H}$. Let $\alpha, \beta \in \mathbb{R}$. If $\alpha < \beta$, then $\mathcal{H}_\beta(F) \subset \mathcal{H}_\alpha(F)$. The inclusion operator

$$i_{\alpha,\beta} \colon \mathcal{H}_\beta(F) \hookrightarrow \mathcal{H}_\alpha(F)$$

is compact with Schmidt representation

$$i_{\alpha,\beta} = \sum_{j=1}^{\infty} \kappa_j^{\beta-\alpha} \left\langle \kappa_j^\beta \varphi_j, \cdot \right\rangle_{\mathcal{H}_\beta} \kappa_j^\alpha \varphi_j.$$

It follows that the s-numbers of the inclusion operator i are $s_j(i) = \kappa_j^{\beta-\alpha}$. In particular, the inclusion operator

$$i_{\alpha,\alpha+1} \colon \mathcal{H}_{\alpha+1}(F) \hookrightarrow \mathcal{H}_\alpha(F)$$

is Hilbert-Schmidt with s-numbers $s_j = \kappa_j$.

Since we shall work in a fixed framed Hilbert space $(\mathcal{H}, F)$, the argument F of the Hilbert spaces $\mathcal{H}_\alpha(F)$ will be often omitted.

Proposition 3.10. *Let $\{A_\iota \in \mathcal{B}(\mathcal{H}), \iota \in I\}$ be a net of bounded operators on a Hilbert space with frame F. The net of operators*

$$|F| A_\iota |F| : \mathcal{H} \to \mathcal{H}$$

converges in $\mathcal{B}(\mathcal{H})$ (respectively, in $\mathcal{L}_p(\mathcal{H})$) if and only if the net of operators

$$A_\iota \colon \mathcal{H}_1 \to \mathcal{H}_{-1}$$

converges in $\mathcal{B}(\mathcal{H}_1(F), \mathcal{H}_{-1}(F))$ (respectively, in $\mathcal{L}_p(\mathcal{H}_1, \mathcal{H}_{-1})$).

Proof. The operators $|F|^{-1} : \mathcal{H}_1 \to \mathcal{H}$ and $|F|^{-1} : \mathcal{H} \to \mathcal{H}_{-1}$ are unitary by Lemma 3.9. Hence, the composition of operators $|F|^{-1} : \mathcal{H}_1 \to \mathcal{H}$, $|F| A_\iota |F| : \mathcal{H} \to \mathcal{H}$ and $|F|^{-1} : \mathcal{H} \to \mathcal{H}_{-1}$ converges in $\mathcal{B}(\mathcal{H}_1(F), \mathcal{H}_{-1}(F))$ (respectively, in $\mathcal{L}_p(\mathcal{H}_1(F), \mathcal{H}_{-1}(F)))$ if

and only if the operator $|F|\, A_\iota\, |F| : \mathcal{H} \to \mathcal{H}$ converges in $\mathcal{B}(\mathcal{H})$ (respectively, in $\mathcal{L}_p(\mathcal{H})$). $\square$

Elements of $\mathcal{H}_1$ are regular (smooth), while elements of $\mathcal{H}_{-1}$ are non-regular. In this sense, the frame operator F increases smoothness of vectors.

Remark 1. *If $\alpha > 0$, then the triple $(\mathcal{H}_\alpha, \mathcal{H}, \mathcal{H}_{-\alpha})$ forms a rigged Hilbert space. So, a frame in a Hilbert space generates a natural rigging. At the same time, a frame evidently contains essentially more information, than a rigging.*

3.5.1. *Diamond conjugate.* Let $\alpha \in \mathbb{R}$. On the product $\mathcal{H}_\alpha \times \mathcal{H}_{-\alpha}$ there exists a unique bounded form $\langle \cdot, \cdot \rangle_{\alpha,-\alpha}$ such that for any $f, g \in \mathcal{H}_{|\alpha|}$

$$\langle f, g \rangle_{\alpha,-\alpha} = \langle f, g \rangle .$$

Let $\mathcal{K}$ be a Hilbert space. For any bounded operator $A\colon \mathcal{H}_\alpha \to \mathcal{K}$, there exists a unique bounded operator $A^\diamondsuit\colon \mathcal{K} \to \mathcal{H}_{-\alpha}$ such that for any $f \in \mathcal{K}$ and $g \in \mathcal{H}_\alpha$ the equality

$$\left\langle A^\diamondsuit f, g \right\rangle_{-\alpha,\alpha} = \langle f, Ag \rangle_{\mathcal{K}}$$

holds. In particular, if $A\colon \mathcal{H}_1 \to \mathcal{K}$ and $f, g \in \mathcal{H}_1$, then

$$\left\langle f, A^\diamondsuit A g \right\rangle_{1,-1} = \langle Af, Ag \rangle_{\mathcal{K}} . \tag{25}$$

There is a connection between the diamond conjugate and usual conjugate

$$A^\diamondsuit = |F|^{-2\alpha} A^*$$

where $A^*\colon \mathcal{K} \to \mathcal{H}_\alpha$ and $|F|^{-2\alpha} : \mathcal{H}_\alpha \to \mathcal{H}_{-\alpha}$. It follows from Lemma 3.9, that if A belongs to $\mathcal{L}_p(\mathcal{H}_\alpha, \mathcal{K})$, then $A^\diamondsuit$ belongs to $\mathcal{L}_p(\mathcal{K}, \mathcal{H}_{-\alpha})$.

3.6. **The trace-class matrix $\varphi(\lambda + iy)$.** Let H_0 be a self-adjoint operator on a framed Hilbert space $(\mathcal{H}, F)$. Let λ be a fixed point of $\Lambda(H_0; F)$. For any $y \geqslant 0$, we introduce the matrix

$$\varphi(\lambda + iy) = \frac{1}{\pi} \left(\kappa_i \kappa_j \left\langle \varphi_i, \operatorname{Im} R_{\lambda+iy}(H_0) \varphi_j \right\rangle \right) \tag{26}$$

and consider it as an operator on ℓ_2.

We note several elementary properties of $\varphi(\lambda + iy)$.

(i) For all $y \geqslant 0$, $\varphi(\lambda + iy)$ is a non-negative trace-class operator on ℓ_2 and its trace is equal to the trace of $\frac{1}{\pi} F \operatorname{Im} R_{\lambda+iy}(H_0) F^*$. This follows from Theorem 2.18 and the fact that $\varphi(\lambda + iy)$ is unitarily equivalent to $\frac{1}{\pi} F \operatorname{Im} R_{\lambda+iy}(H_0) F^*$.

(ii) For all $y > 0$, the kernel of $\varphi(\lambda + iy)$ is trivial.
This follows from the fact that the kernel of $F \operatorname{Im} R_{\lambda+iy}(H_0) F^*$ is trivial. Indeed, otherwise for some non-zero $f \in \mathcal{K}$ $F \operatorname{Im} R_{\lambda+iy}(H_0) F^* f = 0 \Rightarrow \ker R_{\lambda+iy}(H_0) \ni F^* f \neq 0$, which is impossible.

(iii) The matrix $\varphi(\lambda+iy)$ is a real-analytic function of the parameter $y>0$ with values in $\mathcal{L}_1(\ell_2)$, and it is continuous in $\mathcal{L}_1(\ell_2)$ up to $y=0$. This directly follows from Theorem 2.18.

(iv) The estimate
$$s_n(\varphi(\lambda+iy)) \leqslant y^{-1}\kappa_n^2$$
holds. This follows from the equality $s_n(A^*A) = s_n(AA^*)$ and the estimate (11).

3.7. **The Hilbert-Schmidt matrix $\eta(\lambda+iy)$.** Let $\lambda \in \Lambda(H_0;F)$. For any $y \geqslant 0$, we also introduce the matrix

$$\eta(\lambda+iy) = \sqrt{\varphi(\lambda+iy)}. \tag{27}$$

We list elementary properties of $\eta(\lambda+iy)$.

(i) For all $y \geqslant 0$, $\eta(\lambda+iy)$ is a non-negative Hilbert-Schmidt operator on ℓ_2. This follows from 3.6(i).

(ii) If $y>0$, then the kernel of $\eta(\lambda+iy)$ is trivial.
This follows from the fact that the kernel of $\varphi(\lambda+iy)$ is trivial, 3.6(ii).

(iii) The matrix $\eta(\lambda+iy)$ is a real-analytic function of the parameter $y>0$ with values in $\mathcal{L}_2(\mathcal{H})$. This follows from Lemma 2.15 and 3.6(iii).

(iv) The matrix $\eta(\lambda+iy)$ is continuous in $\mathcal{L}_2(\mathcal{H})$ up to $y=0$. This follows from Lemma 2.15 and 3.6(iii).

(v) The estimate
$$s_n(\eta(\lambda+iy)) \leqslant y^{-1/2}\kappa_n$$
holds. This follows from the similar estimate for s-numbers of $\varphi(\lambda+iy)$, see 3.6(iv).

3.8. **Eigenvalues $\alpha_j(\lambda+iy)$ of $\eta(\lambda+iy)$.** Let $\lambda \in \Lambda(H_0;F)$.

We denote by $\alpha_j(\lambda+iy)$ the j-th eigenvalue of $\eta(\lambda+iy)$ (counting multiplicities).

We list elementary properties of $\alpha_j(\lambda+iy)$.

(i) For $y>0$, all eigenvalues $\alpha_j(\lambda+iy)$ are strictly positive. This follows from 3.7(ii).

(ii) For $y \geqslant 0$, the sequence $(\alpha_j(\lambda+iy))$ belongs to ℓ_2. This follows from 3.7(i).

(iii) The functions $(0,\infty) \ni y \mapsto \alpha_j(\lambda+iy)$ can be chosen to be real-analytic (after proper enumeration). This follows from Theorem 2.16 and 3.7(iii).

(iv) All $\alpha_j(\lambda+iy)$ converge as $y \to 0$. This follows from Theorem 2.16 and 3.7(iv).

3.9. **Zero and non-zero type indices.** Let $\lambda \in \Lambda(H_0;F)$. While the eigenvalues $\alpha_j(\lambda+iy)$ of the matrix $\eta(\lambda+iy)$ are strictly positive for $y>0$, the limit values $\alpha_j(\lambda)$ of some of them can be equal to zero. We say that the eigenvalue function $\alpha_j(\lambda+iy)$ is of *non-zero type*, if its limit is not equal to zero. Otherwise we say that it is of *zero type*. We denote the set of non-zero type indices by $\mathcal{Z}_\lambda$.

Though it is not necessary, we agree to enumerate functions $\alpha_j(\lambda+iy)$ in such a way, that the sequence $\{\alpha_j(\lambda+i0)\}$ is decreasing.

3.10. **Vectors $e_j(\lambda+iy)$.** For any $\lambda \in \Lambda(H_0;F)$ we consider the sequence of normalized eigenvectors

$$e_j(\lambda+iy) \in \ell_2, \ j=1,2,\ldots$$

of the non-negative Hilbert-Schmidt matrix $\eta(\lambda+iy)$. These vectors are also eigenvectors of $\varphi(\lambda+iy)$. We enumerate the functions $e_j(\lambda+iy)$ in such a way that

$$\eta(\lambda+iy)e_j(\lambda+iy) = \alpha_j(\lambda+iy)e_j(\lambda+iy), \qquad y>0, \tag{28}$$

where enumeration of $\alpha_j(\lambda+iy)$ is given in subsection 3.9.

We list elementary properties of $e_j(\lambda+iy)$'s.

(i) If $y>0$, then the sequence $e_j(\lambda+iy) \in \ell_2, \ j=1,2,\ldots$ is an orthonormal basis of ℓ_2.
(ii) The functions $(0,\infty) \ni y \mapsto e_j(\lambda+iy) \in \ell_2$ can be chosen to be real-analytic. This follows from Theorem 2.16 and the item 3.7(iii).
(iii) For indices j of non-zero type, the functions $[0,\infty) \ni y \mapsto e_j(\lambda+iy) \in \ell_2$ are continuous up to $y=0$. This follows from Theorem 2.16 and 3.7(iv).
(iv) We say that $e_j(\lambda+iy)$ is of (non-)zero type, if the corresponding eigenvalue function $\alpha_j(\lambda+iy)$ is of (non-)zero type. Non-zero type vectors $e_j(\lambda+iy)$ have limit values $e_j(\lambda+i0)$, which form an orthonormal system in ℓ_2. This follows from Theorem 2.16.
    Note that zero-type vectors $e_j(\lambda+iy)$ may not converge as $y\to 0$.
(v) For non-zero type indices j the vectors $e_j(\lambda+i0)$ are measurable.

3.11. **Vectors $\eta_j(\lambda+iy)$.** Let $\lambda \in \Lambda(H_0;F)$. We introduce the vector $\eta_j(\lambda+iy)$ as the j-th column of the Hilbert-Schmidt matrix $\eta(\lambda+iy)$ (see (27)). This definition implies that

$$\langle \eta_j(\lambda+iy), \eta_k(\lambda+iy)\rangle = \varphi_{jk}(\lambda+iy). \tag{29}$$

We list elementary properties of $\eta_j(\lambda+iy)$'s.

(i) For all $y \geqslant 0$, all vectors $\eta_j(\lambda+iy)$ belong to ℓ_2.
    This is a consequence of the fact that $\eta(\lambda+iy)$ is a bounded operator.
(ii) For all $y \geqslant 0$, the norms of vectors $\eta_j(\lambda+iy)$ constitute a sequence

$$(\|\eta_1(\lambda+iy)\|, \|\eta_2(\lambda+iy)\|, \|\eta_3(\lambda+iy)\|, \ldots),$$

which belongs to ℓ_2. This follows from the fact that $\eta(\lambda+iy)$ is a Hilbert-Schmidt operator for all $y \geqslant 0$ (see 3.7(i) and 3.7(iv)).

(iii) If $y > 0$, then the set of vectors $\{\eta_j(\lambda + iy)\}$ is complete in ℓ_2.
It follows from (28) that

$$\begin{aligned} e_j(\lambda+iy) &= \alpha_j^{-1}(\lambda+iy)\eta(\lambda+iy)e_j(\lambda+iy) \\ &= \alpha_j^{-1}\begin{pmatrix} \eta_{11} & \eta_{12} & \dots \\ \eta_{21} & \eta_{22} & \dots \\ \dots & \dots & \dots \end{pmatrix}\begin{pmatrix} e_{1j} \\ e_{2j} \\ \dots \end{pmatrix} = \alpha_j^{-1}\begin{pmatrix} \eta_{11}e_{1j}+\eta_{12}e_{2j}+\dots \\ \eta_{21}e_{1j}+\eta_{22}e_{2j}+\dots \\ \dots \end{pmatrix} \\ &= \alpha_j^{-1}(\lambda+iy)\sum_{k=1}^{\infty} e_{kj}(\lambda+iy)\eta_k(\lambda+iy), \qquad y \geqslant 0, \end{aligned} \tag{30}$$

where, in case of $y = 0$, the summation is over indices j of non-zero type. Hence, the set of vectors $\{\eta_1(\lambda+iy), \eta_2(\lambda+iy), \dots\}$ is complete. Note also, that the linear combination above is absolutely convergent, according to (ii).

(iv) Let $y > 0$. If for some $\beta = (\beta_j) \in \ell_2$ the equality

$$\sum_{j=1}^{\infty} \beta_j \eta_j(\lambda+iy) = 0$$

holds, then $(\beta_j) = 0$.
Proof. Assume the contrary. We have

$$\begin{aligned} \eta(\lambda+iy)\beta &= \begin{bmatrix} \beta_1\eta_{11}(\lambda+iy) + \beta_2\eta_{12}(\lambda+iy) + \dots \\ \dots \\ \beta_1\eta_{i1}(\lambda+iy) + \beta_2\eta_{i2}(\lambda+iy) + \dots \\ \dots \end{bmatrix} \\ &= \beta_1\eta_1(\lambda+iy) + \beta_2\eta_2(\lambda+iy) + \dots \\ &= 0, \end{aligned}$$

where the second equality makes sense, since the series $\sum_{j=1}^{\infty} \beta_j\eta_j(\lambda+iy)$ is absolutely convergent by 3.11(ii). It follows that β is an eigenvector of $\eta(\lambda+iy)$ corresponding to a zero eigenvalue. Since, by 3.7(ii), for $y > 0$ the matrix $\eta(\lambda+iy)$ does not have zero eigenvalues, we get a contradiction.

(v) Vectors $\eta_j(\lambda+iy)$ converge to $\eta_j(\lambda)$ in ℓ_2 as $y \to 0$. This follows from property 3.7(iv) of $\eta(\lambda+iy)$.

3.12. **Unitary matrix $e(\lambda+iy)$.** Let $\lambda \in \Lambda(H_0; F)$. We can form a matrix

$$e(\lambda+iy) = (e_{jk}(\lambda+iy)),$$

whose columns are $e_j(\lambda+iy)$, $j = 1, 2, \dots$. Since vectors $e_j(\lambda+iy)$, $j = 1, 2, \dots$, form an orthonormal basis of ℓ_2, this matrix is unitary and it diagonalizes the matrix $\eta(\lambda+iy)$:

$$e(\lambda+iy)^*\eta(\lambda+iy)e(\lambda+iy) = \operatorname{diag}(\alpha_1(\lambda+iy), \alpha_2(\lambda+iy), \dots),$$

where $(\alpha_j(\lambda+iy)) \in \ell_2$ are eigenvalues of $\eta(\lambda+iy)$, see subsection 3.8.

3.13. **Vectors $\varphi_j(\lambda+iy)$.** Let $\lambda \in \Lambda(H_0;F)$. Now we introduce vectors

$$\varphi_j(\lambda+iy) = \kappa_j^{-1}\eta_j(\lambda+iy) \in \ell_2. \tag{31}$$

It may seem to be more consistent to denote by $\varphi_j(\lambda+iy)$ the j-th column of the matrix $\varphi(\lambda+iy)$. But, firstly, we don't need columns of $\varphi(\lambda+iy)$, secondly, there is an advantage of this notation. Namely, $\varphi_j(\lambda)$ can be considered as the value of the vector $\varphi_j \in \mathcal{H}$ at $\lambda \in \Lambda(H_0;F)$, as we shall see later (see Section 4).

Some properties of $\varphi_j(\lambda+iy)$.

(i) All vectors $\varphi_j(\lambda+iy)$ belong to ℓ_2. This follows from $\eta_j(\lambda+iy) \in \ell_2$, see 3.11(i).

(ii) If $y>0$, then the set of vectors $\{\varphi_j(\lambda+iy)\}$ is complete in ℓ_2. This follows from similar property of $\{\eta_j(\lambda+iy)\}$, see 3.11(iii).

(iii) Let $y>0$. If $(\kappa_j^{-1}\beta_j) \in \ell_2$ and

$$\sum_j \beta_j\varphi_j(\lambda+iy) = 0,$$

then $(\beta_j)=0$. This follows from similar property of $\eta_j(\lambda+iy)$ (see 3.11(iv)) and (31).

(iv) The following equality holds

$$\langle \varphi_j(\lambda+iy), \varphi_k(\lambda+iy)\rangle = \frac{1}{\pi}\left\langle \varphi_j, \operatorname{Im} R_{\lambda+iy}(H_0)\varphi_k\right\rangle. \tag{32}$$

Indeed, using (31), (29) and (26),

$$\begin{aligned}\langle \varphi_j(\lambda+iy), \varphi_k(\lambda+iy)\rangle &= \kappa_j^{-1}\kappa_k^{-1}\left\langle \eta_j(\lambda+iy), \eta_k(\lambda+iy)\right\rangle \\ &= \kappa_j^{-1}\kappa_k^{-1}\varphi_{jk}(\lambda+iy) \\ &= \frac{1}{\pi}\left\langle \varphi_j, \operatorname{Im} R_{\lambda+iy}(H_0)\varphi_k\right\rangle.\end{aligned}$$

(v) It follows from (30) and (31), that each $e_j(\lambda+iy)$ can be written as a linear combination of $\varphi_j(\lambda+iy)$'s with coefficients of the form $\kappa_j\beta_j$, where $(\beta_j) \in \ell_2$:

$$e_j(\lambda+iy) = \alpha_j^{-1}(\lambda+iy)\sum_{k=1}^{\infty}\kappa_k e_{kj}(\lambda+iy)\varphi_k(\lambda+iy). \tag{33}$$

Moreover, this representation is unique, according to (iii).

(vi) For all $j=1,2,\ldots \quad \|\varphi_j(\lambda+iy)\|_{\ell_2} \leqslant (y\pi)^{-1/2}$.
Indeed, by (32),

$$\|\varphi_j(\lambda+iy)\|^2 = \frac{1}{\pi}\left\langle \varphi_j, \operatorname{Im} R_{\lambda+iy}(H_0)\varphi_j\right\rangle \leqslant \frac{1}{y\pi}.$$

(vii) $\varphi_j(\lambda+iy)$ converges to $\varphi_j(\lambda)$ in ℓ_2, as $y\to 0$. This follows from 3.11(v).

(viii) The equality

$$\langle \varphi_j(\lambda), \varphi_k(\lambda)\rangle_{\ell_2} = \frac{1}{\pi}\left\langle \varphi_j, \operatorname{Im} R_{\lambda+i0}(H_0)\varphi_k\right\rangle_{\mathcal{H}}$$

holds.

Proof. Since for $\lambda \in \Lambda(H_0;F)$ the limit on the right hand side exists by 3.6(iii), this follows from (vii) and (iv).

3.14. **The operator $\mathcal{E}_{\lambda+iy}$.** Let $\lambda \in \Lambda(H_0;F)$. Let

$$\mathcal{E}_{\lambda+iy} \colon \mathcal{H}_1 \to \ell_2$$

be a linear operator defined on the frame vectors by the formula

$$\mathcal{E}_{\lambda+iy}\varphi_j = \varphi_j(\lambda+iy). \tag{34}$$

Some properties of $\mathcal{E}_{\lambda+iy}$.

(i) For $y>0$, the equality

$$\langle \mathcal{E}_{\lambda+iy}\varphi_j, \mathcal{E}_{\lambda+iy}\varphi_k\rangle_{\ell_2} = \frac{1}{\pi}\langle \varphi_j, \operatorname{Im} R_{\lambda+iy}(H_0)\varphi_k\rangle_{\mathcal{H}}$$

holds. This is immediate from the definition of $\mathcal{E}_{\lambda+iy}$ and (32).

It follows that

$$\mathcal{E}^*_{\lambda+iy}\mathcal{E}_{\lambda+iy} = \frac{1}{\pi}\operatorname{Im} R_{\lambda+iy}(H_0). \tag{35}$$

(ii) Let $y>0$. The operator $\mathcal{E}_{\lambda+iy}$ is a Hilbert-Schmidt operator as an operator from $\mathcal{H}_1$ to ℓ_2. Moreover,

$$\|\mathcal{E}_{\lambda+iy}\|^2_{\mathcal{L}_2(\mathcal{H}_1,\ell_2)} = \frac{1}{\pi}\operatorname{Tr}_{\mathcal{K}}\left(F \operatorname{Im} R_{\lambda+iy}(H_0)F^*\right).$$

Indeed, evaluating the trace of $\mathcal{E}^*_{\lambda+iy}\mathcal{E}_{\lambda+iy}$ in the orthonormal basis $\{\kappa_j\varphi_j\}$ of $\mathcal{H}_1$, we get, using (i) and (21),

$$\begin{aligned}
\sum_{j=1}^\infty \left\langle \mathcal{E}^*_{\lambda+iy}\mathcal{E}_{\lambda+iy}\kappa_j\varphi_j, \kappa_j\varphi_j\right\rangle_{\mathcal{H}_1} &= \sum_{j=1}^\infty \kappa_j^2\langle \mathcal{E}_{\lambda+iy}\varphi_j, \mathcal{E}_{\lambda+iy}\varphi_j\rangle_{\ell_2} \\
&= \frac{1}{\pi}\sum_{j=1}^\infty \kappa_j^2\langle \varphi_j, \operatorname{Im} R_{\lambda+iy}(H_0)\varphi_j\rangle_{\mathcal{H}} && \text{by (i)} \\
&= \frac{1}{\pi}\sum_{j=1}^\infty \langle F^*\psi_j, \operatorname{Im} R_{\lambda+iy}(H_0)F^*\psi_j\rangle_{\mathcal{H}} && \text{by (21)} \\
&= \frac{1}{\pi}\operatorname{Tr}_{\mathcal{K}}(F\operatorname{Im} R_{\lambda+iy}(H_0)F^*). && \text{by (12)}
\end{aligned}$$

(iii) The norm of $\mathcal{E}_{\lambda+iy}\colon \mathcal{H}_1 \to \ell_2$ is $\leqslant \|\eta(\lambda+iy)\|_2$. Indeed, if $\beta = (\beta_j) \in \ell_2$, then $f := \sum_{j=1}^\infty \kappa_j\beta_j\varphi_j \in \mathcal{H}_1$ with $\|f\|_{\mathcal{H}_1} = \|\beta\|$, and, using (34), (31) and Schwarz

inequality, one gets

$$\begin{aligned}\|\mathcal{E}_{\lambda+iy}f\| &= \left\|\sum_{j=1}^{\infty}\kappa_j\beta_j\varphi_j(\lambda+iy)\right\| = \left\|\sum_{j=1}^{\infty}\beta_j\eta_j(\lambda+iy)\right\| \\ &\leqslant \|\beta\|\cdot\left(\sum_{j=1}^{\infty}\|\eta_j(\lambda+iy)\|^2\right)^{1/2} = \|f\|_{\mathcal{H}_1}\cdot\|\eta(\lambda+iy)\|_2\,.\end{aligned}$$

(iv) For all $y>0$, the operator $\mathcal{E}_{\lambda+iy}\colon\mathcal{H}_1\to\ell_2$ has trivial kernel.
Indeed, otherwise for some non-zero vector $f\in\mathcal{H}_1$,

$$0=\langle\mathcal{E}_{\lambda+iy}f,\mathcal{E}_{\lambda+iy}f\rangle=\frac{1}{\pi}\langle f,\operatorname{Im}R_{\lambda+iy}(H_0)f\rangle\,.$$

Combining this equality with the formula

$$\operatorname{Im}R_{\lambda+iy}(H_0)=yR_{\lambda-iy}(H_0)R_{\lambda+iy}(H_0),$$

one infers that $R_{\lambda+iy}(H_0)$ has non-trivial kernel. But this is impossible.

(v) The operator $\mathcal{E}_{\lambda+iy}\colon\mathcal{H}_1\to\ell_2$ as a function of $y>0$ is real-analytic in $\mathcal{L}_2(\mathcal{H}_1,\ell_2)$.

(vi) The operator $\mathcal{E}_{\lambda+iy}\colon\mathcal{H}_1\to\ell_2$ converges in the Hilbert-Schmidt norm to $\mathcal{E}_\lambda$, as $y\to 0$.
Proof. We have, in the orthonormal basis $\{\kappa_j\varphi_j\}$ of $\mathcal{H}_1$,

$$\begin{aligned}\|\mathcal{E}_{\lambda+iy}-\mathcal{E}_\lambda\|^2_{\mathcal{L}_2(\mathcal{H}_1)} &= \sum_{j=1}^{\infty}\|(\mathcal{E}_{\lambda+iy}-\mathcal{E}_\lambda)(\kappa_j\varphi_j)\|^2 && \text{by (13)} \\ &= \sum_{j=1}^{\infty}\|\kappa_j\varphi_j(\lambda+iy)-\kappa_j\varphi_j(\lambda)\|^2 && \text{by (34)} \\ &= \sum_{j=1}^{\infty}\|\eta_j(\lambda+iy)-\eta_j(\lambda)\|^2 && \text{by (31)} \\ &= \|\eta(\lambda+iy)-\eta(\lambda)\|_2^2\to 0 && \text{by (13)}\end{aligned}$$

by 3.7(iv).

(vii) It follows that the equality in (i) holds for $y=0$ as well

$$\langle\mathcal{E}_\lambda\varphi_j,\mathcal{E}_\lambda\varphi_k\rangle_{\ell_2}=\frac{1}{\pi}\langle\varphi_j,\operatorname{Im}R_{\lambda+i0}(H_0)\varphi_k\rangle_{\mathcal{H}}\,.$$

Moreover, the operator $\mathcal{E}_\lambda\colon\mathcal{H}_1\to\ell_2$ is also Hilbert-Schmidt and

$$\|\mathcal{E}_\lambda\|^2_{\mathcal{L}_2(\mathcal{H}_1,\ell_2)}=\frac{1}{\pi}\operatorname{Tr}_{\mathcal{K}}(F\operatorname{Im}R_{\lambda+i0}(H_0)F^*)\,.$$

3.15. **Vectors $b_j(\lambda+iy)\in\mathcal{H}_1$.** Let $y>0$. For each $j=1,2,3\ldots$ we introduce the vector $b_j(\lambda+iy)\in\mathcal{H}_1$ as a unique vector from the Hilbert space $\mathcal{H}_1$ with property

$$\mathcal{E}_{\lambda+iy}b_j(\lambda+iy)=e_j(\lambda+iy). \tag{36}$$

Property 3.13(v) of $\varphi_j(\lambda+iy) = \mathcal{E}_{\lambda+iy}\varphi_j$ implies that the vector

$$b_j(\lambda+iy) = \alpha_j^{-1}(\lambda+iy)\sum_{k=1}^{\infty}\kappa_k e_{kj}(\lambda+iy)\varphi_k \tag{37}$$

satisfies the above equation, where $e_{kj}(\lambda+iy)$ is the k's coordinate of $e_j(\lambda+iy)$. Property 3.13(iii) of $\varphi_j(\lambda+iy)$ implies that such representation is unique.

The representation (37) shows that the functions $(0,\infty) \ni y \mapsto b_j(\lambda+iy) \in \mathcal{H}_1$ are continuous, since, by Schwarz inequality and $\|e_j(\lambda+iy)\| = 1$, the series in the right hand side of (37) absolutely converges locally uniformly with respect to $y > 0$.

We list some properties of the vectors $b_j(\lambda+iy)$.

(i) The relations

$$\|b_j(\lambda+iy)\|_{\mathcal{H}} \leqslant \alpha_j^{-1}(\lambda+iy)\,\|F\|_2\,,$$
$$\|b_j(\lambda+iy)\|_{\mathcal{H}_1} = \alpha_j^{-1}(\lambda+iy)$$

hold. The inequality follows from (37) and Schwarz inequality. The equality follows from the definition of $\mathcal{H}_1$-norm and from $\|e_j(\lambda+iy)\| = 1$.

(ii) Vectors $b_j(\lambda+iy)$, $j = 1,2,\ldots$, are linearly independent.
Indeed, this is because the vectors $\mathcal{E}_{\lambda+iy}b_j(\lambda+iy) = e_j(\lambda+iy)$, $j = 1,2,\ldots$, are linearly independent.

(iii) The system of vectors $\{b_j(\lambda+iy)\}$ is complete in $\mathcal{H}_1$ (and, consequently, in $\mathcal{H}$ as well).
Proof. This follows from the equality

$$\varphi_l = \kappa_l^{-1}\sum_{j=1}^{\infty}\bar{e}_{lj}(\lambda+iy)\alpha_j(\lambda+iy)b_j(\lambda+iy), \quad l = 1,2,\ldots \tag{38}$$

This equality itself follows from (37) and from the unitarity of the matrix $(e_{jk}(\lambda+iy))$.

(iv) The equality

$$\langle \mathcal{E}_{\lambda+iy}b_j(\lambda+iy), \mathcal{E}_{\lambda+iy}b_k(\lambda+iy)\rangle = \delta_{jk} \tag{39}$$

holds.
This immediately follows from the definition of $b_j(\lambda+iy)$'s and the fact that the system $\{e_j(\lambda+iy)\}$ is an orthonormal basis in ℓ_2 (see item 3.10(i)).

(v) The equality

$$\frac{y}{\pi}\langle R_{\lambda\pm iy}(H_0)b_j(\lambda+iy), R_{\lambda\pm iy}(H_0)b_k(\lambda+iy)\rangle = \delta_{jk}$$

holds.

Proof. We have

$$
\begin{aligned}
&\frac{y}{\pi}\left\langle R_{\lambda+iy}(H_0)b_j(\lambda+iy), R_{\lambda+iy}(H_0)b_k(\lambda+iy)\right\rangle \\
&\qquad = \frac{y}{\pi}\left\langle b_j(\lambda+iy), R_{\lambda-iy}(H_0)R_{\lambda+iy}(H_0)b_k(\lambda+iy)\right\rangle \\
&\qquad = \left\langle b_j(\lambda+iy), \frac{1}{\pi}\operatorname{Im} R_{\lambda+iy}(H_0)b_k(\lambda+iy)\right\rangle \\
&\qquad = \left\langle b_j(\lambda+iy), \mathcal{E}^*_{\lambda+iy}\mathcal{E}_{\lambda+iy}b_k(\lambda+iy)\right\rangle && \text{by (35)} \\
&\qquad = \left\langle \mathcal{E}_{\lambda+iy}b_j(\lambda+iy), \mathcal{E}_{\lambda+iy}b_k(\lambda+iy)\right\rangle \\
&\qquad = \delta_{jk}. && \text{by (39)}
\end{aligned}
$$

(vi) The set of vectors $\sqrt{\frac{y}{\pi}}\left\{R_{\lambda+iy}(H_0)b_j(\lambda+iy)\right\}$ is an orthonormal basis in $\mathcal{H}$.
Proof. By (v), it is enough to show that this set is complete. If for a non-zero vector g

$$\langle R_{\lambda+iy}(H_0)b_j(\lambda+iy), g\rangle = 0$$

for all j, then

$$\langle b_j(\lambda+iy), R_{\lambda-iy}(H_0)g\rangle = 0$$

for all j. By completeness (iii) of the set $\{b_j(\lambda+iy)\}$, one infers from this that $R_{\lambda-iy}(H_0)g = 0$. This is impossible, since $R_{\lambda-iy}(H_0)$ has trivial kernel.

(vii) If j is of non-zero type, then $b_j(\lambda+iy) \in \mathcal{H}_1$ converges in $\mathcal{H}_1$ to $b_j(\lambda+i0) \in \mathcal{H}_1$. This follows from the convergence of $e_j(\lambda+iy)$ in ℓ_2 (see item 3.10(iii)) and (37).

4. Construction of the direct integral

As it was mentioned before, a frame in a Hilbert space $\mathcal{H}$, on which a self-adjoint operator H_0 acts, allows to define explicitly the fiber Hilbert space $\mathfrak{h}_\lambda$ of the direct integral of Hilbert spaces diagonalizing H_0, with the purpose to define $f(\lambda)$ as an element of $\mathfrak{h}_\lambda$ for a dense set $\mathcal{H}_1$ of vectors and any λ from a fixed set of full Lebesgue measure $\Lambda(H_0; F)$. In this section we give this construction.

Let H_0 be a self-adjoint operator on a fixed framed Hilbert space $(\mathcal{H}, F)$, where the frame F is given by (20). For $\lambda \in \Lambda(H_0; F)$ (see Definition 3.2), we have a Hilbert-Schmidt operator (see item 3.14(vi))

$$\mathcal{E}_\lambda \colon \mathcal{H}_1 \to \ell_2,$$

defined by the formula

$$\mathcal{E}_\lambda f = \sum_{j=1}^{\infty} \beta_j \eta_j(\lambda), \tag{40}$$

where $f = \sum\limits_{j=1}^{\infty} \beta_j \kappa_j \varphi_j \in \mathcal{H}_1$, $(\beta_j) \in \ell_2$ (see item (v) of subsection 3.11 for definition of $\eta_j(\lambda)$). The formula (40) is one of the most key formulas in this paper. Since, by 3.11(ii),

$(\|\eta_j(\lambda)\|) \in \ell_2$, the series above converges absolutely: by the Schwarz inequality

$$\sum_{j=1}^{\infty} \|\beta_j \eta_j(\lambda)\|_{\ell_2} \leqslant \|\beta\|_{\ell_2} \left(\sum_{j=1}^{\infty} \|\eta_j(\lambda)\|_{\ell_2}^2 \right)^{1/2} = \|\beta\|_{\ell_2} \|\eta\|_2 .$$

The set $\mathcal{E}_\lambda \mathcal{H}_1$ is a pre-Hilbert space. We denote the closure of this set in ℓ_2 by $\mathfrak{h}_\lambda$:

$$\mathfrak{h}_\lambda := \overline{\mathcal{E}_\lambda \mathcal{H}_1}. \tag{41}$$

It is clear that the dimension function $\Lambda(H_0; F) \ni \lambda \mapsto \dim \mathfrak{h}_\lambda$ is Borel measurable, since, by definition,

$$\dim \mathfrak{h}_\lambda = \operatorname{rank}(\eta(\lambda)) \in \{0, 1, 2, \ldots, \infty\},$$

and it's clear that the matrix $\eta(\lambda)$ is Borel measurable. Since the matrix $\varphi(\lambda)$ is self-adjoint, it is also clear that

$$\dim \mathfrak{h}_\lambda = \operatorname{rank}(\varphi(\lambda)).$$

One can give one more formula for $\dim(\lambda)$

$$\operatorname{Card}\{j \colon j \text{ is of non-zero type}\} = \dim \mathfrak{h}_\lambda.$$

Lemma 4.1. *The system of vector-functions $\{\varphi_j(\lambda), j = 1, 2, \ldots\}$ satisfies the axioms of the measurability base (Definition 2.11) for the family of Hilbert spaces $\{\mathfrak{h}_\lambda\}_{\lambda \in \Lambda(H_0;F)}$, given by (41).*

Proof. For any fixed $\lambda \in \Lambda(H_0; F)$, vectors $\varphi_1(\lambda), \varphi_2(\lambda), \ldots$ generate $\mathfrak{h}_\lambda$ by definition. Measurability of functions $\Lambda(H_0; F) \ni \lambda \mapsto \langle \varphi_j(\lambda), \varphi_k(\lambda) \rangle$ follows from 3.13(viii). So, both axioms of the measurability base hold. $\square$

The field of Hilbert spaces

$$\{\mathfrak{h}_\lambda \colon \lambda \in \Lambda(H_0; F)\}$$

with measurability base

$$\lambda \mapsto \mathcal{E}_\lambda \varphi_j = \varphi_j(\lambda), \quad j = 1, 2, \ldots \tag{42}$$

determines a direct integral of Hilbert spaces (see subsection 2.6)

$$\mathcal{H} := \int_{\Lambda(H_0;F)}^{\oplus} \mathfrak{h}_\lambda \, d\lambda. \tag{43}$$

The vector $\varphi_j(\lambda)$ is to be interpreted as the value of the vector φ_j at λ, as we shall see later. Note that though the vectors $\varphi_j(\lambda) \in \mathfrak{h}_\lambda$, $j = 1, 2, \ldots$ depend on the sequence (κ_j) of weights of the frame F, their norms and scalar products

$$\|\varphi_j(\lambda)\|_{\mathfrak{h}_\lambda}, \quad \langle \varphi_j(\lambda), \varphi_k(\lambda) \rangle$$

are independent of weights, as directly follows from 3.13(viii). This also means that if two frames F_1 and F_2 have different weights, but the same frame vectors, and if λ belongs to both full sets $\Lambda(H_0, F_1)$ and $\Lambda(H_0, F_2)$, then the Hilbert spaces $\mathfrak{h}_\lambda(H_0, F_1)$ and $\mathfrak{h}_\lambda(H_0, F_2)$ are naturally isomorphic. The isomorphism is given by the correspondence

$$\varphi_j^{(1)}(\lambda) \longleftrightarrow \varphi_j^{(2)}(\lambda), \ j = 1, 2, \ldots,$$

where $\varphi_j^{(k)}(\lambda)$, $k=1,2$, is the vector constructed using the frame F_k.

Example 4.2. *Let $\mathcal{H} = L^2(\mathbb{T}) \ominus \{\text{constants}\}$ and let*

$$F = \sum_{j \in \mathbb{Z}_*} |j|^{-1} e^{ij\lambda},$$

where $\mathbb{Z}_* = \{\pm 1, \pm 2, \ldots\}$. Let H_0 be the multiplication by λ. In this case

$$\varphi(\lambda) = \left(|jk|^{-1} e^{i(j-k)\lambda}\right)_{j,k \in \mathbb{Z}_*}$$

and $\Lambda(H_0; F) = \mathbb{R}$. For all $\lambda \in \mathbb{R}$, this matrix has rank one, so that there is only one index of non-zero type and $\dim \mathfrak{h}_\lambda = 1$. This corresponds to the fact that H_0 has simple spectrum. Vectors f from $\mathcal{H}_1$ are absolutely continuous functions with L^2 derivative. The value of φ_j at λ should be interpreted as the jth column of $\eta(\lambda) = \sqrt{\varphi(\lambda)}$ over $|j|$. The matrix $\eta(\lambda)$ is difficult to calculate. For the only non-zero type index 1 we have $\alpha_1(0)^2 = 2\sum_{n=1}^{\infty} n^{-2}$.

Lemma 4.3. *For any $j = 1, 2, \ldots$, the function $\mathcal{E}\varphi_j$ belongs to $\mathcal{H}$ and $\|\mathcal{E}\varphi_j\|_{\mathcal{H}} \leqslant 1$.*

Proof. We only need to show that $\varphi_j(\lambda) = \mathcal{E}_\lambda \varphi_j$ is square summable and that the estimate holds. It follows from 3.13(viii) and (21) that

$$\begin{aligned} \langle \mathcal{E}\varphi_j, \mathcal{E}\varphi_j \rangle_{\mathcal{H}} &= \int_{\Lambda(H_0;F)} \langle \varphi_j(\lambda), \varphi_j(\lambda) \rangle \, d\lambda \\ &= \frac{1}{\pi} \int_{\Lambda(H_0;F)} \langle \varphi_j, \operatorname{Im} R_{\lambda+i0}(H_0)\varphi_j \rangle \, d\lambda =: (E). \end{aligned}$$

Since $\frac{1}{\pi}\langle \varphi_j, \operatorname{Im} R_{\lambda+iy}(H_0)\varphi_j \rangle$ is the Poisson integral of the function $\left\langle E_\lambda^{H_0}\varphi_j, \varphi_j \right\rangle$, it follows from Theorem 2.3 that

$$\frac{1}{\pi}\langle \varphi_j, \operatorname{Im} R_{\lambda+i0}(H_0)\varphi_j \rangle = \frac{d}{d\lambda}\left\langle E_\lambda^{H_0}\varphi_j, \varphi_j \right\rangle$$

for a.e. λ. Consequently,

$$(E) = \int_{\Lambda(H_0;F)} \frac{d}{d\lambda}\left\langle \varphi_j, E_\lambda^{H_0}\varphi_j \right\rangle d\lambda \leqslant 1.$$

□

Corollary 4.4. *For any pair of indices j and k the function*

$$\Lambda(H_0; F) \ni \lambda \mapsto \langle \varphi_j(\lambda), \varphi_k(\lambda) \rangle_{\mathfrak{h}_\lambda}$$

is summable and its L_1-norm is $\leqslant 1$.

Proof. This follows from Schwarz inequality and Lemma 4.3. □

A function $\Lambda(H_0; F) \ni \lambda \to f(\lambda) \in \ell_2$ will be called $\mathcal{H}$-*measurable,* if $f(\lambda) \in \mathfrak{h}_\lambda$ for a.e. $\lambda \in \Lambda(H_0; F)$, if $f(\cdot)$ is measurable with respect to the measurability base (42) and if $f \in \mathcal{H}$ (i.e. if f is square summable).

We can define a linear operator $\mathcal{E}\colon \mathcal{H}_1 \to \mathcal{H}$ with dense domain $\mathcal{D}$, by the formula

$$(\mathcal{E}\varphi_j)(\lambda) = \varphi_j(\lambda), \tag{44}$$

where $\mathcal{D}$ is defined by (23).

One can define a standard minimal core $\mathcal{A}(H_0, F)$ of the absolutely continuous spectrum of H_0, acting on a framed Hilbert space, by the formula

$$\mathcal{A}(H_0, F) = \bigcup_{i,j=1}^{\infty} \mathcal{A}(m_{ij}),$$

where $m_{ij}(\Delta) = \langle E_\Delta \varphi_i, \varphi_j \rangle$ is a (signed) spectral measure, and $\mathcal{A}(m)$ is a standard minimal support of m.

Proposition 4.5. *The dimension of the fiber Hilbert space $\mathfrak{h}_\lambda$ is not zero if and only if $\lambda \in \mathcal{A}(H_0, F)$.*

Proof. ($\Leftarrow$). If $\lambda \in \mathcal{A}(H_0, F)$, then for some pair (i, j) of indices $\lambda \in \mathcal{A}(m_{ij})$. This means that the limit $\mathcal{C}_{m_{ij}}(\lambda + i0)$ exists and is not zero. This implies that $\varphi(\lambda) = (\varphi_{ij}(\lambda)) \neq 0$, as well as $\eta(\lambda) \neq 0$. So, the Hilbert space $\mathfrak{h}_\lambda$ is generated by at least one non-zero vector $\varphi_j(\lambda)$.

($\Rightarrow$). If $\dim \mathfrak{h}_\lambda \neq 0$, then by definition (41) of $\mathfrak{h}_\lambda$ for some index j the vector $\varphi_j(\lambda) = \kappa_j^{-1}\eta_j(\lambda)$ is non-zero. Hence, the matrix $\eta(\lambda)$ is non-zero. It follows that $\varphi(\lambda)$ is non-zero. If $\varphi_{ij}(\lambda) \neq 0$, then $\lambda \in \mathcal{A}(m_{ij})$. So, $\lambda \in \mathcal{A}(H_0, F)$. □

It follows from this Proposition that the direct integral (43) can be rewritten as

$$\mathcal{H} = \int_{\mathcal{A}(H_0,F)}^{\oplus} \mathfrak{h}_\lambda \, d\lambda. \tag{45}$$

Hence, instead of the full set $\Lambda(H_0; F)$ one can use $\mathcal{A}(H_0, F)$. However, since the set $\Lambda(H_0; F)$ has full Lebesgue measure, it is more convenient to work with.

Recall that the vectors $e_j(\lambda), j = 1, 2, \ldots$, corresponding to non-zero type indices j, are the limit values of the non-zero type eigenvectors $e_j(\lambda + iy), j = 1, 2, \ldots$ of $\eta(\lambda + iy) = \sqrt{\varphi(\lambda + iy)}$.

Lemma 4.6. *The system of ℓ_2-vectors $\{e_j(\lambda)\colon j$ is of non-zero type$\}$ is an orthonormal basis of $\mathfrak{h}_\lambda$.*

Proof. Firstly, by 3.10(iv), the system of vectors $\{e_j(\lambda)\colon j \text{ is of non-zero type}\}$ is orthonormal. In part (A) it is shown that this system is a subset of $\mathfrak{h}_\lambda$; in part (B) it is shown that the system is complete in $\mathfrak{h}_\lambda$.

(A) By definition (41) of $\mathfrak{h}_\lambda$, it is generated by $\{\varphi_1(\lambda), \varphi_2(\lambda), \ldots\}$, or, which is the same, by $\{\eta_1(\lambda), \eta_2(\lambda), \ldots\}$. For a non-zero type index j, one can take the limit $y \to 0^+$

in (30) to get

$$e_j(\lambda) = \alpha_j(\lambda)^{-1} \sum_{k=0}^{\infty} e_{kj}(\lambda)\eta_k(\lambda).$$

It follows that $\{e_j(\lambda)\colon j \text{ is of non-zero type}\} \subset \mathfrak{h}_\lambda$.

(B) For any index i the following formula holds

$$\eta_i(\lambda+iy) = \sum_{k=1}^{\infty} \alpha_k(\lambda+iy)e_{ik}(\lambda+iy)e_k(\lambda+iy). \tag{46}$$

Indeed, this equality is equivalent to the following one

$$\langle \eta_i(\lambda+iy), e_j(\lambda+iy)\rangle = \alpha_j(\lambda+iy)e_{ij}(\lambda+iy).$$

This equality follows from (28). Passing to the limit in (46), one gets

$$\eta_i(\lambda) = \sum_{k=1, k\in\mathcal{Z}_\lambda}^{\infty} \alpha_k(\lambda)e_{ik}(\lambda)e_k(\lambda).$$

It follows that the system $\{e_j(\lambda), j \text{ is of non-zero type}\}$ is complete in $\mathfrak{h}_\lambda$. □

This lemma implies that $\{e_j(\lambda)\}$ is an orthonormal measurability base for the direct integral $\mathcal{H}$.

Let $P_\lambda \in \mathcal{B}(\ell_2)$ be the projection onto $\mathfrak{h}_\lambda$.

Lemma 4.7. *There exists a measurable operator-valued function $\Lambda(H_0; F) \ni \lambda \mapsto \psi(\lambda) \in \mathcal{C}(\ell_2)$ such that $\psi(\lambda)$ is a self-adjoint operator and*

$$\psi(\lambda)\varphi(\lambda) = P_\lambda.$$

Proof. Since $\varphi(\lambda)$ is a non-negative compact operator, this follows from the spectral theorem. We just set $\psi(\lambda) = 0$ on $\ker\varphi(\lambda)$ and $\psi(\lambda) = \varphi(\lambda)^{-1}$ on $\ker\varphi(\lambda)^{\perp}$. □

Corollary 4.8. *The family of orthogonal projections $P_\lambda\colon \ell_2 \to \mathfrak{h}_\lambda$ is weakly measurable.*

Proof. This follows from Lemmas 3.5 and 4.7. □

Lemma 4.9. *A function $f\colon \Lambda(H_0; F) \ni \lambda \mapsto f(\lambda) \in \mathfrak{h}_\lambda$ is $\mathcal{H}$-measurable if and only if it is measurable as a function $\Lambda(H_0; F) \to \ell_2$ and is square summable.*

Proof. (If) Since the functions $\varphi_j(\lambda)$ are measurable, if a function $f\colon \Lambda(H_0; F) \to \ell_2$ is measurable and $f(\lambda) \in \mathfrak{h}_\lambda$, then all the functions $\langle f(\lambda), \varphi_j(\lambda)\rangle_{\mathfrak{h}_\lambda} = \langle f(\lambda), \varphi_j(\lambda)\rangle_{\ell_2}$ are measurable. Hence, f is $\mathcal{H}$-measurable.

(Only if) Let $f(\lambda) \in \mathfrak{h}_\lambda$ be $\mathcal{H}$-measurable, i.e. be such that for any j

$$\langle \varphi_j(\lambda), f(\lambda)\rangle$$

is measurable and $\|f(\lambda)\|_{\mathfrak{h}_\lambda} \in L^2(\Lambda, d\lambda)$. This implies that the vector

$$(\kappa_j \langle \varphi_j(\lambda), f(\lambda)\rangle) = (\langle \eta_j(\lambda), f(\lambda)\rangle) = \eta(\lambda)f(\lambda)$$

is measurable. So, the function $\eta^2(\lambda)f(\lambda) = \varphi(\lambda)f(\lambda)$ is also measurable. Since by Lemma 4.7 there exists a measurable function $\psi(\lambda)$, such that $\psi(\lambda)\varphi(\lambda) = P_\lambda$, the function $f(\lambda)$ is also measurable. ☐

Proposition 4.10. *Let $\chi_\Delta(\Delta)$ be the characteristic function of Δ. The set of finite linear combinations of functions*

$$\Lambda(H_0; F) \ni \lambda \mapsto \chi_\Delta(\lambda)\varphi_j(\lambda) \in \ell_2,$$

where Δ is an arbitrary Borel subset of Λ and $j = 1, 2, \ldots,$ is dense in $\mathcal{H}$.

Proof. This follows from Lemma 2.14. ☐

4.1. $\mathcal{E}$ **is an isometry.** Note that the system $\left\{\varphi_j^{(a)}\right\}$ is complete in $\mathcal{H}^{(a)}$ though it is, in general, not linearly independent.

Proposition 4.11. *Let H_0 be a self-adjoint operator on a framed Hilbert space $(\mathcal{H}, F)$. The operator $\mathcal{E}\colon \mathcal{H}_1 \to \mathcal{H}$, defined by (44), is bounded as an operator from $\mathcal{H}$ to $\mathcal{H}$, so that one can define $\mathcal{E}$ on the whole $\mathcal{H}$ by continuity. The operator $\mathcal{E}\colon \mathcal{H} \to \mathcal{H}$, thus defined, vanishes on $\mathcal{H}^{(s)}$ and is isometric on $\mathcal{H}^{(a)}$.*

Proof. Firstly, we show that $\mathcal{E}$ is bounded. It follows from the item 3.13(viii) that

$$\begin{aligned}\langle \mathcal{E}\varphi_j, \mathcal{E}\varphi_k\rangle_{\mathcal{H}} &= \int_\Lambda \langle \mathcal{E}_\lambda\varphi_j, \mathcal{E}_\lambda\varphi_k\rangle_{\mathfrak{h}_\lambda}\, d\lambda \\ &= \int_\Lambda \langle \varphi_j(\lambda), \varphi_k(\lambda)\rangle_{\mathfrak{h}_\lambda}\, d\lambda = \frac{1}{\pi}\int_\Lambda \langle \varphi_j, \operatorname{Im} R_{\lambda+i0}(H_0)\,\varphi_k\rangle\, d\lambda.\end{aligned}$$

Since by Theorem 2.3

$$\frac{1}{\pi}\langle \varphi_j, \operatorname{Im} R_{\lambda+i0}(H_0)\varphi_k\rangle = \frac{d}{d\lambda}\langle \varphi_j, E_\lambda\varphi_k\rangle \quad \text{for a.e. } \lambda \in \Lambda, \tag{47}$$

it follows that

$$\langle \mathcal{E}\varphi_j, \mathcal{E}\varphi_k\rangle_{\mathcal{H}} = \int_\Lambda \frac{d\langle \varphi_j, E_\lambda\varphi_k\rangle}{d\lambda}\, d\lambda.$$

This implies that

$$\langle \mathcal{E}\varphi_j, \mathcal{E}\varphi_k\rangle_{\mathcal{H}} = \int_\Lambda \frac{d\langle \varphi_j, E_\lambda^{(a)}\varphi_k\rangle}{d\lambda}\, d\lambda = \langle \varphi_j, E_\Lambda^{(a)}\varphi_k\rangle = \langle \varphi_j^{(a)}, \varphi_k^{(a)}\rangle. \tag{48}$$

This equality implies that for any $f \in \mathcal{D}$ (see (23) for the definition of $\mathcal{D}$) $\|\mathcal{E}f\| = \left\|f^{(a)}\right\| \leqslant \|f\|$, and so, $\mathcal{E}$ is bounded. Since also $\|\mathcal{E}f\| = \left\|P^{(a)}f\right\|$ for all f from the dense set $\mathcal{D}$, it follows that for any $f \in \mathcal{H}$ $\|\mathcal{E}f\| = \left\|P^{(a)}f\right\|$. This implies that $\mathcal{E}$ vanishes on $\mathcal{H}^{(s)}$ and it is an isometry on $\mathcal{H}^{(a)}$. The proof is complete. ☐

This Proposition implies that for any $f \in \mathcal{H}$ we have a vector-function $f(\lambda) = \mathcal{E}_\lambda(f)$ as an element of the direct integral (43). The function $f(\lambda)$ is defined for a.e. $\lambda \in \Lambda$, while for regular vectors $f \in \mathcal{H}_1$ $f(\lambda)$ is defined for all $\lambda \in \Lambda(H_0; F)$.

Lemma 4.12. *For any $f, g \in \mathcal{H}^{(a)}$ the equality*

$$\langle f, g \rangle = \int_\Lambda \langle f(\lambda), g(\lambda) \rangle \, d\lambda$$

holds.

Proof. Indeed, the right hand side of this equality is, by definition, $\langle \mathcal{E} f, \mathcal{E} g \rangle_{\mathfrak{H}}$, which by (48) is equal to $\langle f, g \rangle_{\mathcal{H}}$. □

4.2. $\mathcal{E}$ **is a unitary.** The aim of this subsection is to show that the restriction of the operator $\mathcal{E} \colon \mathcal{H} \to \mathfrak{H}$ to $\mathcal{H}^{(a)}$ is unitary.

Lemma 4.13. *Let Δ be a Borel subset of $\Lambda = \Lambda(H_0; F)$. If $f \in E_{\Lambda \setminus \Delta} \mathcal{H}$, then $f(\lambda)$ is equal to zero on Δ for a.e. $\lambda \in \Delta$.*

Proof. (A) If $g = \sum\limits_{j=1}^{N} \alpha_j \varphi_j \in \mathcal{D}$ (see (23)), then $\|E_\Delta g\|^2 = \int_\Delta \langle g(\lambda), g(\lambda) \rangle \, d\lambda$.
Proof of (A).

$$\begin{aligned}
\int_\Delta \langle g(\lambda), g(\lambda) \rangle \, d\lambda &= \sum_{j=1}^{N} \sum_{k=1}^{N} \bar\alpha_j \alpha_k \int_\Delta \langle \varphi_j(\lambda), \varphi_k(\lambda) \rangle \, d\lambda \\
&= \sum_{j=1}^{N} \sum_{k=1}^{N} \bar\alpha_j \alpha_k \int_\Delta \frac{d}{d\lambda} \langle \varphi_j, E_\lambda \varphi_k \rangle \, d\lambda && \text{by Thm. 2.4} \\
&= \sum_{j=1}^{N} \sum_{k=1}^{N} \bar\alpha_j \alpha_k \int_\Delta \frac{d}{d\lambda} \left\langle \varphi_j, E_\lambda^{(a)} \varphi_k \right\rangle \, d\lambda && \text{by Cor. 2.7} \\
&= \sum_{j=1}^{N} \sum_{k=1}^{N} \bar\alpha_j \alpha_k \left\langle \varphi_j, E_\Delta^{(a)} \varphi_k \right\rangle \\
&= \left\| E_\Delta^{(a)} g \right\|^2 .
\end{aligned}$$

Since $\Delta \subset \Lambda(H_0; F)$, it follows from Corollary 3.8 that $E_\Delta^{(a)} = E_\Delta$. It follows that $\left\| E_\Delta^{(a)} g \right\|^2 = \|E_\Delta g\|^2$.

(B) Proof of the lemma. Note that $f \in E_{\Lambda \setminus \Delta} \mathcal{H}$ implies that f is an absolutely continuous vector for H_0. Consequently, there exists a sequence $f_1, f_2, \ldots$ of vectors from $P^{(a)} \mathcal{D}$ converging to f (in $\mathcal{H}$). Then by Lemma 4.12

$$\int_{\Lambda(H_0; F)} \left\langle f(\lambda) - f^N(\lambda), f(\lambda) - f^N(\lambda) \right\rangle \, d\lambda = \left\| f - f^N \right\|^2 \to 0.$$

Since by (A)

$$\int_\Delta \left\langle f^N(\lambda), f^N(\lambda) \right\rangle \, d\lambda = \left\| E_\Delta f^N \right\|^2 = \left\| E_\Delta (f - f^N) \right\|^2 \leqslant \left\| f - f^N \right\|^2 \to 0,$$

it follows that

$$\int_{\Delta} \langle f(\lambda), f(\lambda) \rangle \, d\lambda = 0.$$

So, $f(\lambda) = 0$ for a.e. $\lambda \in \Delta$. □

Corollary 4.14. *Let Δ be a Borel subset of $\Lambda(H_0, F)$ and let $f, g \in \mathcal{H}$. If $E_\Delta f = E_\Delta g$, then $f(\lambda) = g(\lambda)$ for a.e. $\lambda \in \Delta$.*

Corollary 4.15. *For any Borel subset Δ of $\Lambda(H_0; F)$ and any $f \in \mathcal{H}$*

$$\mathcal{E}(E_\Delta f)(\lambda) = \chi_\Delta(\lambda) f(\lambda) \qquad \text{a.e. } \lambda \in \mathbb{R}.$$

Corollary 4.16. *Let Δ be a Borel subset of $\Lambda(H_0; F)$. For any $f, g \in \mathcal{H}$,*

$$\langle E_\Delta f, E_\Delta g \rangle = \int_{\Delta} \langle f(\lambda), g(\lambda) \rangle \, d\lambda.$$

Proposition 4.17. *The map $\mathcal{E} \colon \mathcal{H}^{(a)} \to \mathfrak{H}$ is unitary.*

Proof. It has already been proven (Proposition 4.11) that $\mathcal{E}$ is an isometry with initial space $\mathcal{H}^{(a)}$. So, it is enough to show that the range of $\mathcal{E}$ coincides with $\mathfrak{H}$. Corollary 4.15 implies that the range of $\mathcal{E}$ contains all functions of the form $\chi_\Delta(\cdot)\varphi_j(\cdot)$, where Δ is an arbitrary Borel subset of $\Lambda(H_0; F)$ and $j = 1, 2, 3 \ldots$ Consequently, Proposition 4.10 completes the proof. □

4.3. Diagonality of H_0 in $\mathfrak{H}$.

The aim of this subsection is to prove Theorem 4.19, which asserts that the direct integral $\mathfrak{H}$ is a spectral representation of $\mathcal{H}$ for the operator $H_0^{(a)}$.

Using standard step-function approximation argument, it follows from Corollary 4.15 that

Theorem 4.18. *For any bounded Borel function h on $\Lambda(H_0; F)$ and any $f \in \mathcal{H}$*

$$\mathcal{E}_\lambda(h(H_0)f) = h(\lambda)\mathcal{E}_\lambda f \qquad \text{for a.e. } \lambda \in \Lambda.$$

This theorem implies the following result.

Theorem 4.19. *$H_0^{(a)}$ is naturally isomorphic to the operator of multiplication by λ on $\mathfrak{H}$ via the unitary mapping $\mathcal{E} \colon \mathcal{H}^{(a)} \to \mathfrak{H}$:*

$$\mathcal{E}_\lambda(H_0 f) = \lambda \mathcal{E}_\lambda f \qquad \text{for a.e. } \lambda \in \mathbb{R}.$$

Nonetheless, we give another proof of this theorem.

Lemma 4.20. [Y, (1.3.12)] *Let H be a self-adjoint operator on Hilbert space $\mathcal{H}$, and let $f, g \in \mathcal{H}$. Then for a.e. $\lambda \in \mathbb{R}$*

$$\lambda \frac{d}{d\lambda} \langle f, E_\lambda g \rangle = \frac{d}{d\lambda} \langle H_0 f, E_\lambda g \rangle,$$

Proof of Theorem 4.19. It is enough to show that for any $f \in E_\Delta \mathcal{H}$, and for a.e. $\lambda \in \Delta$ the equality $\mathcal{E}_\lambda(H_0 f) = \lambda f(\lambda)$ holds, where Δ is any bounded Borel subset of Λ.

This is equivalent to the statement: for any $g \in E_\Delta \mathcal{H}$

$$\int_\Delta \langle \mathcal{E}_\lambda(H_0 f), g(\lambda) \rangle \, d\lambda = \int_\Delta \lambda \, \langle f(\lambda), g(\lambda) \rangle \, d\lambda.$$

By continuity of $H_0 E_\Delta^{H_0}$ and of the multiplicator $\lambda \chi_\Delta(\lambda)$, it is enough to consider the case of $f = E_\Delta \varphi_j \in \mathcal{H}^{(a)}$ and $g = E_\Delta \varphi_k \in \mathcal{H}^{(a)}$. Then, by (47) and Corollary 4.14, the right hand side of the previous formula is

$$\int_\Delta \lambda \frac{d}{d\lambda} \langle \varphi_j, E_\lambda \varphi_k \rangle \, d\lambda = \int_\Delta \frac{d}{d\lambda} \langle H_0 \varphi_j, E_\lambda \varphi_k \rangle \, d\lambda = \langle H_0 \varphi_j, E_\Delta \varphi_k \rangle \,,$$

where Lemma 4.20 has been used. Now, Corollary 4.16 completes the proof. □

A complete set of unitary invariants of the absolutely continuous part $H_0^{(a)}$ of the operator H_0 is given by the sequence $(\Lambda_0, \Lambda_1, \Lambda_2, \ldots)$, where

$$\Lambda_n = \{\lambda \in \Lambda(H_0; F) \colon \dim \mathfrak{h}_\lambda = n\} \,.$$

One of the versions of the spectral theorem says that there exists a direct integral representation

$$\mathcal{H}^{(a)} \cong \int_{\hat\sigma}^{\oplus} \mathfrak{h}_\lambda \, \rho(d\lambda),$$

of the Hilbert space $\mathcal{H}^{(a)}$, which diagonalizes $H_0^{(a)}$, where $\hat\sigma$ is a core of the spectrum of H_0, and ρ is a measure from the spectral type of H_0. Actually, instead of changing the measure ρ in its spectral type, it is possible to change (renormalize) the scalar product of the fiber Hilbert spaces $\mathfrak{h}_\lambda$. In the construction of the direct integral, given in this section, a frame in $\mathcal{H}$ in particular fixes a renormalization of scalar products in fiber Hilbert spaces.

The operator $\mathcal{E}_\lambda$ is the evaluation operator which answers the question (18). As we have seen, for any vector $f \in \mathcal{H}_1$ and any point λ of the set of full Lebesgue measure $\Lambda(H_0; F)$, one can define the value of the vector f at λ by the formula

$$f(\lambda) = \mathcal{E}_\lambda f.$$

Vectors f, which belong to $\mathcal{H}_1$, can be defined at every point of the set $\Lambda(H_0; F)$, since they are regular; or, rather the contrary, vectors of $\mathcal{H}_1$ should be considered regular, since they can be defined at every point of $\Lambda(H_0; F)$. If a vector f is not regular, that is, if $f \notin \mathcal{H}_1$, then one can define its value only at almost every point of $\Lambda(H_0; F)$. Results of this section fully justify this interpretation of the operator $\mathcal{E}_\lambda$.

Remark 2. Recall that a vector f is called cyclic for a self-adjoint operator H_0, if vectors $H_0^k f$, $k = 0, 1, 2, \ldots$ generate the whole Hilbert space $\mathcal{H}$. The construction of the direct integral obviously implies that if H_0 has a cyclic vector then $\dim \mathfrak{h}_\lambda \leqslant 1$ for all $\lambda \in \Lambda(H_0; F)$.

Remark 3. *Clearly, the family* $\Omega_1 := \{e_j(\lambda)\}$ *is a measurability base and it generates the same set of measurable vector-functions as the measurability base* $\Omega_0 := \{\varphi_j(\lambda)\}$*; that is* $\hat{\Omega}_0 = \hat{\Omega}_1$*. The family* Ω_1 *is an orthonormal measurability base.*

5. The resonance set $R(\lambda; \{H_r\}, F)$

In the previous section we have defined the evaluation operator $\mathcal{E}_\lambda$. The evaluation operator is defined on the set $\Lambda(H_0; F)$. Since eventually the operator H_0 is going to be perturbed, one needs to investigate what happens to the set $\Lambda(H_0; F)$ when H_0 is perturbed. Clearly, the complement of $\Lambda(H_0; F)$ consists of points where the operator H_0 behaves in some sense badly. Indeed, by Corollary 3.8 the set $\mathbb{R} \setminus \Lambda(H_0; F)$ is a core of the singular spectrum of H_0. So, one of the reasons, for which a vector $f \in \mathcal{H}$ cannot be defined at some point $\lambda \in \mathbb{R}$ is that λ can be an eigenvalue of H_0.

The results of this section are generally well-known (cf. e.g. [Ar, Ag, SW, S_3, Y]). I do not claim any originality for them.

So far we have considered a single fixed self-adjoint operator H_0 on a Hilbert space $\mathcal{H}$ with a frame F. Now we are going to perturb H_0 by self-adjoint trace-class operators.

We say that an operator-function $\mathbb{R} \ni r \mapsto A(r)$ is piecewise analytic in appropriate norm, if it is continuous in the norm, and if there is a finite or infinite increasing sequence of numbers $r_j,\ j \in \mathbb{Z}$ with no finite accumulation points, such that the restriction of $A(r)$ to any interval $[r_{j-1}, r_j]$ has analytic continuation in the norm to a neighbourhood of that interval.

Given a frame $F \in \mathcal{L}_2(\mathcal{H}, \mathcal{K})$ in a Hilbert space $\mathcal{H}$, we introduce a vector space $\mathcal{A}(F)$ of trace-class operators by

$$\mathcal{A}(F) = \{FJF^* \colon J \in \mathcal{B}(\mathcal{K})\}. \tag{49}$$

For an operator $FJF^* \in \mathcal{A}(F)$ we define its norm by

$$\|FJF^*\|_{\mathcal{A}(F)} = \|J\|.$$

Obviously, the vector space $\mathcal{A}(F)$ with such a norm is a Banach space.

Assumption 5.1. *Let* $F \colon \mathcal{H} \to \mathcal{K}$ *be a frame operator in a Hilbert space* $\mathcal{H}$*. We assume that the path*

$$\mathbb{R} \ni r \mapsto H_r$$

of self-adjoint operators in $\mathcal{H}$ *satisfies the following conditions:*
(i) $H_r = H_0 + V_r$,
(ii) $V_r = F^* J_r F$, *where* J_r *is a bounded self-adjoint operator on the Hilbert space* $\mathcal{K}$,
(iii) the path

$$\mathbb{R} \ni r \mapsto J_r \in \mathcal{B}(\mathcal{K})$$

is piecewise real-analytic.

In other words, $H_r \in H_0 + \mathcal{A}(F)$ and the path $\{H_r\}$ is $\mathcal{A}(F)$-analytic.

Clearly, $V_0 = 0$. Obviously, the path $\{V_r\}$ is piecewise real-analytic with values in $\mathcal{L}_1(\mathcal{K})$, so that the trace-class derivative

$$\dot{V}_r = F^* \dot{J}_r F$$

exists and it is trace-class. Since the derivative $\dot{V}_r$ belongs to $\mathcal{A}(F)$, it can be considered as an operator $\mathcal{H}_{-1} \to \mathcal{H}_1$. Clearly, $\dot{V}_r$ satisfies the following condition:

$$\dot{V}_r \colon \mathcal{H}_{-1} \to \mathcal{H}_1 \quad \text{is a bounded operator.} \tag{50}$$

Assumption 5.1 is not too restrictive.

Lemma 5.2. *Let H be a self-adjoint operator in $\mathcal{H}$ and let V be a self-adjoint trace-class operator in $\mathcal{H}$. There exists a frame $F \in \mathcal{L}_2(\mathcal{H}, \mathcal{K})$ and a path $\{H_r\}$ which satisfies Assumption 5.1, such that $H_0 = H$ and $H_1 = H + V$.*

Proof. Let $H_r = H + rV$ and $\mathcal{K} = \mathcal{H}$. If V has trivial kernel, then one can take

$$F = \sqrt{|V|},$$

so that $V = F^* \operatorname{sign}(V) F$. If V has non-trivial kernel, then one can take $F = \sqrt{|V|} + I \cdot \tilde{F}$, where I is the projection onto $\ker(V)$ and $\tilde{F}$ is a self-adjoint Hilbert-Schmidt operator on the Hilbert space $I\mathcal{H}$. □

Let

$$T_r(z) = T(z, r) = F R_z(H_r) F^*.$$

Lemma 5.3. *If operators $A_\alpha, A \in \mathcal{B}(\mathcal{H})$ are invertible and $A_\alpha \to A$ uniformly, then $A_\alpha^{-1} \to A^{-1}$ uniformly.*

Proof. Since

$$A_\alpha^{-1} - A^{-1} = A_\alpha^{-1} \left(A - A_\alpha\right) A^{-1},$$

it is enough to show that eventually $\|A_\alpha^{-1}\| \leqslant \text{const}$. Note that

$$\left\|A^{-1}\right\| = \sup_{f \neq 0} \frac{\|A^{-1} f\|}{\|f\|} = \sup_{g \neq 0} \frac{\|g\|}{\|Ag\|} = \left(\inf_{g \neq 0} \frac{\|Ag\|}{\|g\|}\right)^{-1}.$$

So, we need to show that eventually

$$\inf_{\|g\|=1} \|A_\alpha g\| \geqslant c > 0.$$

For this we write

$$\|A_\alpha g\| \geqslant \|Ag\| - \|(A - A_\alpha) g\|$$

and observe that for some $c > 0$ and for all unit length g $\|Ag\| \geqslant c$ and that eventually $\|(A - A_\alpha) g\| < \frac{c}{2}$. □

The following lemma and its proof are well-known (cf. e.g. [Ag, Theorem 4.2], [Y, Lemma 4.7.8]). They are given for completeness.

Lemma 5.4. *The operator $1 + J_r T_0(z)$ is invertible for all $r \in \mathbb{R}$ and all $z \in \mathbb{C} \setminus \mathbb{R}$.*

Proof. (A) The equality (Aronszajn's equation [Ar], cf. also [SW, S_3])

$$T_r(z)(1 + J_r T_0(z)) = T_0(z) \tag{51}$$

holds. Proof.

The second resolvent identity

$$R_z(H_r) - R_z(H_0) = -R_z(H_r) V_r R_z(H_0) \tag{52}$$

implies

$$R_z(H_r)(1 + V_r R_z(H_0)) = R_z(H_0).$$

Using Assumption 5.1(ii) and multiplying this equality by F from the left and by F^* from the right, we get (51).

(B) Since $T_0(z)$ is compact, if $1 + J_r T_0(z)$ is not invertible, then there exists a non-zero $\psi \in \mathcal{K}$, such that

$$(1 + J_r T_0(z))\psi = 0. \tag{53}$$

Combining this equality with (51) gives $T_0(z)\psi = 0$. Combining this equality with (53) gives $\psi = 0$. This contradiction shows that $1 + J_r T_0(z)$ is invertible. $\square$

While the operator $1 + J_r T_0(z)$ is invertible for all non-real values of z, the operator $1 + J_r T_0(\lambda + i0)$ may not be invertible at some points. The set of points where $1 + J_r T_0(\lambda + i0)$ is not invertible is of special importance.

Definition 5.5. *Let $\{H_r\}$ be a path of self-adjoint operators on $\mathcal{H}$ with frame F, which satisfies Assumption 5.1. Let $\lambda \in \Lambda(H_0; F)$. We denote by $R(\lambda; \{H_r\}, F)$ the set*

$$R(\lambda; \{H_r\}, F) := \{r \in \mathbb{R} \colon 1 + J_r T_0(\lambda + i0) \text{ is not invertible}\} \tag{54}$$

and call it the resonance set *at λ.*

Lemma 5.6. *The set $R(\lambda; \{H_r\}, F)$ is discrete, i.e. it has no finite accumulation points.*

Proof. Since V_r is a piecewise analytic function, this directly follows from the analytic Fredholm alternative (Theorem 2.17). $\square$

Lemma 5.7. *Let $\lambda \in \mathbb{R}$ be such that the limit $T_0(\lambda + i0)$ exists in the Hilbert-Schmidt norm. Then the limit $T_r(\lambda + i0)$ exists in the Hilbert-Schmidt norm if and only if $r \notin R(\lambda; \{H_r\}, F)$.*

Proof. (Only if) Assume that $T_r(\lambda + i0)$ exists. Taking the Hilbert-Schmidt norm limit $y = \operatorname{Im} z \to 0$ in (51), one gets

$$T_r(\lambda + i0)(1 + J_r T_0(\lambda + i0)) = T_0(\lambda + i0). \tag{55}$$

Since T_0 is compact, $1 + J_r T_0(\lambda + i0)$ is not invertible if and only if there exists a non-zero $\psi \in \mathcal{H}$, such that $(1 + J_r T_0(\lambda + i0))\psi = 0$. This and (55) imply that $T_0(\lambda + i0)\psi = 0$. Hence $\psi = 0$. This contradiction shows that $1 + J_r T_0(\lambda + i0)$ is invertible.

(If) By (51) and Lemma 5.4,

$$T_r(\lambda+iy) = T_0(\lambda+iy)\Big[1 + J_r T_0(\lambda+iy)\Big]^{-1}. \tag{56}$$

If $1 + J_r T_0(\lambda + i0)$ is invertible, then by Lemma 5.3 the limit of the right hand side as $y \to 0^+$ exists in the Hilbert-Schmidt norm. □

Theorem 5.8. *Let $\{H_r\}$ be a path of self-adjoint operators on $\mathcal{H}$ with frame F, which satisfies Assumption 5.1. Let $\lambda \in \Lambda(H_0; F)$. For all $r \notin R(\lambda; \{H_r\}, F)$ the inclusion $\lambda \in \Lambda(H_r; F)$ holds, where $R(\lambda; \{H_r\}, F)$ is a discrete subset of $\mathbb{R}$, defined in (54).*

Proof. (A) Since $\lambda \in \Lambda(H_0, F)$, the limit $T_0(\lambda + i0)$ exists in the Hilbert-Schmidt norm. It follows from Lemma 5.7, that the limit of

$$T_r(\lambda+iy) = F R_{\lambda+iy}(H_r) F^*$$

exists in the Hilbert-Schmidt norm as well.

Now, in order to prove that $\lambda \in \Lambda(H_r, F)$, we need to show that the limit of $F \operatorname{Im} R_{\lambda+iy}(H_r) F^*$ exists in $\mathcal{L}_1$-norm.

(B) The formula

$$\operatorname{Im} T_r(z) = (1 + T_0(\bar z) J_r)^{-1} \operatorname{Im} T_0(z)\, (1 + J_r T_0(z))^{-1} \tag{57}$$

holds.

Proof. Using (56), one has

$$\begin{aligned}
\operatorname{Im} T_r(z) &= \frac{1}{2i}\left(T_r(z) - T_r^*(z)\right) \\
&= \frac{1}{2i}\left(T_0(z)\Big[1 + J_r T_0(z)\Big]^{-1} - \Big[1 + T_0(\bar z) J_r\Big]^{-1} T_0(\bar z)\right) \\
&= \frac{1}{2i}(1 + T_0(\bar z) J_r)^{-1}\left(\Big[1 + T_0(\bar z) J_r\Big] T_0(z) - T_0(\bar z)\Big[1 + J_r T_0(z)\Big]\right)(1 + J_r T_0(z))^{-1} \\
&= \frac{1}{2i}(1 + T_0(\bar z) J_r)^{-1}\left(T_0(z) - T_0(\bar z)\right)(1 + J_r T_0(z))^{-1} \\
&= (1 + T_0(\bar z) J_r)^{-1} \operatorname{Im} T_0(z)\, (1 + J_r T_0(z))^{-1}.
\end{aligned}$$

(C) Since $r \notin R(\lambda; \{H_r\}, F)$, it follows from Lemmas 5.3 and 5.4 that

$$(1 + T_0(\bar z) J_r)^{-1} \quad \text{and} \quad (1 + J_r T_0(z))^{-1}$$

converge in $\|\cdot\|$ as $y = \operatorname{Im} z \to 0^+$. Since, by definition of $\Lambda(H_0; F)$, $\operatorname{Im} T_0(z)$ converges to $\operatorname{Im} T_0(\lambda + i0)$ in $\mathcal{L}_1(\mathcal{K})$, it follows from (57) that $\operatorname{Im} T_r(z)$ also converges in $\mathcal{L}_1(\mathcal{K})$ as $\operatorname{Im} z \to 0^+$. Hence, $\lambda \in \Lambda(H_r; F)$.

That $R(\lambda; \{H_r\}, F)$ is a discrete subset of $\mathbb{R}$ follows from Lemma 5.6. □

Theorem 5.8 shows that the resonance subset of the plane (λ, r) behaves differently with respect to the spectral parameter λ and with respect to the coupling constant r. While

for a fixed r the resonance set is a more or less arbitrary null set, and, consequently, can be very bad, for a fixed λ the resonance set is a discrete subset of $\mathbb{R}$.

The discreteness property of the resonance set $R(\lambda; \{H_r\}, F)$ for a.e. λ is used in an essential way in subsection 8.2.

Proposition 5.9. *If $\lambda \in \Lambda(H_0; F)$ is an eigenvalue of H_r, then $r \in R(\lambda; \{H_r\}, F)$.*

Proof. Since, by Corollary 3.8, the complement of $\Lambda(H_r; F)$ is a support of the singular spectrum of H_r, which includes all eigenvalues of H_r, it follows that if $\lambda \in \Lambda(H_0; F)$ is an eigenvalue of H_r, then $\lambda \notin \Lambda(H_r; F)$, so that by Theorem 5.8 $r \in R(\lambda; \{H_r\}, F)$. □

This proposition partly explains why elements of $R(\lambda; \{H_r\}, F)$ are called resonance points. Note that the inclusion $r \in R(\lambda; \{H_r\}, F)$ does not necessarily imply that λ is an eigenvalue of H_r.

Theorem 5.10. *Let $\lambda \in \Lambda(H_0; F)$. Then $\lambda \notin \Lambda(H_r; F)$ if and only if $r \in R(\lambda; \{H_r\}, F)$.*

Proof. The only if part has been established in Theorem 5.8. The if part says that $\lambda \in \Lambda(H_r; F)$ implies $r \notin R(\lambda; \{H_r\}, F)$. This follows from Lemma 5.7. □

Remark 4. As can be seen from the proofs, existence of $T_0(\lambda+i0)$ in $\mathcal{L}_2(\mathcal{K})$ or existence of $\operatorname{Im} T_0(\lambda+i0)$ in $\mathcal{L}_1(\mathcal{K})$ is not essential for the above theorem. The ideals $\mathcal{L}_2(\mathcal{K})$ and $\mathcal{L}_1(\mathcal{K})$ can be replaced by any $\mathcal{L}_p(\mathcal{K})$, $p \in [1, \infty]$, or even by any invariant operator ideal. What the last theorem is saying is that, as long as r_0 is not a resonance point, the regularity of λ is the same for $r = 0$ and $r = r_0$.

6. Wave matrix $w_\pm(\lambda; H_r, H_0)$

In the main setting of the abstract scattering theory, which considers trace-class perturbations V of arbitrary self-adjoint operators H_0, one first shows existence of the wave operators (Kato-Rosenblum theorem, [Ka, R], cf. also [Y, §6.2])

$$W_\pm(H_1, H_0)\colon \mathcal{H}^{(a)}(H_0) \to \mathcal{H}^{(a)}(H_1),$$

where $H_1 = H_0 + V$, and after that one shows existence of the wave matrices

$$w_\pm(\lambda; H_1, H_0)\colon \mathfrak{h}_\lambda(H_0) \to \mathfrak{h}_\lambda(H_1) \tag{58}$$

for almost every $\lambda \in \mathbb{R}$, where $\mathfrak{h}_\lambda(H_j)$ is a fiber Hilbert space from a direct integral, diagonalizing the absolutely continuous parts $H_j^{(a)}$, $j = 1, 2$, of the operators H_j. A drawback of this definition is that, for a given point $\lambda \in \mathbb{R}$, it is not possible to say whether $w_\pm(\lambda; H_1, H_0)$ is defined or not. This is because the fiber Hilbert spaces $\mathfrak{h}_\lambda(H_j)$ are not explicitly defined: they exist for almost every λ, but for a fixed λ the space $\mathfrak{h}_\lambda(H_j)$ is not defined.

But if we fix a frame F in the Hilbert space $\mathcal{H}$, then for $\lambda \in \Lambda(H_0; F) \cap \Lambda(H_1; F)$ it becomes possible to define the wave matrices $w_\pm(\lambda; H_1, H_0)$ as operators (58),

where $\mathfrak{h}_\lambda(H_j)$, $j = 1, 2$, are the fiber Hilbert spaces associated with the fixed frame by (41).

While the original proof of Kato and Rosenblum used time-dependent methods, the method of this paper is based on the stationary approach to abstract scattering theory from [BE, Y]. Combination of ideas from [BE, Y] with the construction of the direct integral, given in section 4, allows to define wave matrices $w_\pm(\lambda; H_r, H_0)$ for all λ from the set of full Lebesgue measure $\Lambda(H_0; F) \cap \Lambda(H_r; F)$ and prove all their main properties, including the multiplicative property.

In this section H_0 is a self-adjoint operator on $\mathcal{H}$ with frame F, V is a trace-class self-adjoint operator, which satisfies the condition (50). We note again, that for any trace-class self-adjoint operator V there exists a frame F, such that (50) holds. Consequently, the condition (50) does not impose any additional restrictions on the perturbation V, except the trace-class condition.

6.1. **Operators $\mathfrak{a}_\pm(\lambda; H_r, H_0)$.** In [Ag], instead of sandwiching the resolvent, it is considered as acting on appropriately defined Hilbert spaces. Following this idea, we consider the limit value $R_{\lambda+i0}(H_0)$ of the resolvent as an operator

$$R_{\lambda+i0}(H_0) \colon \mathcal{H}_1 \to \mathcal{H}_{-1}.$$

Recall that all Hilbert spaces $\mathcal{H}_\alpha$, $\alpha \in \mathbb{R}$ are naturally isomorphic with the isomorphism being

$$|F|^{\beta-\alpha} : \mathcal{H}_\alpha \to \mathcal{H}_\beta$$

So, if we have an operator-function $A(y), y > 0$, with values in some subclass of $\mathcal{B}(\mathcal{H})$, such that the limit

$$\lim_{y\to 0} |F|^\alpha A(y) |F|^\beta$$

exists in the topology of that class, then the limit

$$\lim_{y\to 0} A(y)$$

exists in the topology of the corresponding subclass of $\mathcal{B}(\mathcal{H}_\beta, \mathcal{H}_{-\alpha})$. In this way we write $A(0)$, meaning by this an operator from $\mathcal{H}_\beta$ to $\mathcal{H}_{-\alpha}$. It is not necessary to use this convention, but otherwise we would need to write a lot of F's in the subsequent formulas, thus making them cumbersome.

Thus, in an expression such as

$$R_{\lambda\mp iy}(H_0)V_r$$

with $y > 0$, both operators $R_{\lambda\mp iy}(H_0)$ and V_r can be understood as operators from $\mathcal{H}$ to $\mathcal{H}$, *or* the operator V_r can be understood as an operator from $\mathcal{H}_{-1}$ to $\mathcal{H}_1$ and the operator $R_{\lambda\mp iy}(H_0)$ can be understood as an operator from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$. But when we take the limit $y \to 0$ and write

$$R_{\lambda\mp i0}(H_0)V_r$$

both operators should be understood in the second sense, so that the product above is an operator from $\mathcal{H}_{-1}$ to $\mathcal{H}_{-1}$. That is, in the product the operator $V_r \colon \mathcal{H}_{-1} \to \mathcal{H}_1$ means actually the operator $|F| V_r |F|$, acting in the following way:

$$\mathcal{H}_1 \xleftarrow{|F|} \mathcal{H} \xleftarrow{V_r} \mathcal{H} \xleftarrow{|F|} \mathcal{H}_{-1}.$$

In the Hilbert space $\mathcal{H}$ the operator $R_{\lambda\mp i0}(H_0)V_r$ (if one wishes) should be written as

$$|F|\, R_{\lambda\mp i0}(H_0)\, |F|\, V_r,$$

where V_r is understood as acting from $\mathcal{H}$ to $\mathcal{H}$.

In the sequel we constantly use this convention without further reference.

Lemma 6.1. *If $\lambda \in \Lambda(H_r; F)$, then*

$$R_{\lambda\pm iy}(H_r) \to R_{\lambda\pm i0}(H_r)$$

in $\mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_{-1})$ as $y \to 0^+$.

Proof. This follows from Theorem 2.19 and Proposition 3.10. □

Lemma 6.2. *If $\lambda \in \Lambda(H_r; F)$, then*

$$\operatorname{Im} R_{\lambda+iy}(H_r) \to \operatorname{Im} R_{\lambda+i0}(H_r)$$

in $\mathcal{L}_1(\mathcal{H}_1, \mathcal{H}_{-1})$ as $y \to 0^+$.

Proof. This follows from Theorem 2.18 and Proposition 3.10. □

We now investigate the forms $\mathfrak{a}_\pm(H_r, H_0; f, g; \lambda)$ (cf. [Y, Definition 2.7.2]). Unlike [Y, Definition 2.7.2], we treat $\mathfrak{a}_\pm(H_r, H_0; \lambda)$ not as a form, but as an operator from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$. In [Y, §5.2] it is proved that this form is well-defined for a.e. $\lambda \in \mathbb{R}$. In the next proposition we give an explicit set of full measure on which $\mathfrak{a}_\pm(H_r, H_0; \lambda)$ exists.

Proposition 6.3. *If $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$, then the limit*

$$\lim_{y\to 0^+} \frac{y}{\pi} R_{\lambda\mp iy}(H_r) R_{\lambda\pm iy}(H_0) \tag{59}$$

exists in $\mathcal{L}_1(\mathcal{H}_1, \mathcal{H}_{-1})$.

Proof. We have (cf. e.g. [Y, (2.7.10)])

$$\begin{aligned} &\frac{y}{\pi} R_{\lambda\mp iy}(H_r) R_{\lambda\pm iy}(H_0) \\ &\qquad = \frac{1}{\pi} \operatorname{Im} R_{\lambda+iy}(H_r) \Big[1 + V_r R_{\lambda\pm iy}(H_0)\Big] \\ &\qquad = \Big[1 - R_{\lambda\mp iy}(H_r) V_r\Big] \cdot \frac{1}{\pi} \operatorname{Im} R_{\lambda+iy}(H_0). \end{aligned} \tag{60}$$

Since $\lambda \in \Lambda(H_0; F) \cap \Lambda(H_r; F)$, by Lemma 6.2, the limits of $\operatorname{Im} R_{\lambda+iy}(H_0)$ and $\operatorname{Im} R_{\lambda+iy}(H_r)$ exist in $\mathcal{L}_1(\mathcal{H}_1, \mathcal{H}_{-1})$. Also, by Lemma 6.1, the limits of $R_{\lambda\pm iy}(H_0)$ and $R_{\lambda\pm iy}(H_r)$ exist in $\mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_{-1})$, while $V \colon \mathcal{H}_{-1} \to \mathcal{H}_1$ is a bounded operator (see (50)). It follows that the limit (59) exists in $\mathcal{L}_1(\mathcal{H}_1, \mathcal{H}_{-1})$. □

Definition 6.4. *Let $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$. The operators*

$$\mathfrak{a}_\pm(\lambda; H_r, H_0) \colon \mathcal{H}_1 \to \mathcal{H}_{-1}$$

are the limits (59) taken in $\mathcal{L}_1(\mathcal{H}_1, \mathcal{H}_{-1})$ topology.

Proposition 6.5. *If $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$, then, in $\mathcal{L}_1(\mathcal{H}_1, \mathcal{H}_{-1})$, the equalities*

$$\begin{aligned} \mathfrak{a}_\pm(\lambda; H_r, H_0) &= \Big[1 - R_{\lambda\mp i0}(H_r)V_r\Big] \cdot \frac{1}{\pi} \operatorname{Im} R_{\lambda+i0}(H_0) \\ &= \frac{1}{\pi} \operatorname{Im} R_{\lambda+i0}(H_r) \Big[1 + V_r R_{\lambda\pm i0}(H_0)\Big] \end{aligned} \tag{61}$$

hold.

Proof. This follows from (60), Lemmas 6.1, 6.2, Proposition 6.3 and (50). □

Note that products such as $R_{\lambda\mp i0}(H_r)V \cdot \frac{1}{\pi} \operatorname{Im} R_{\lambda+i0}(H_0)$ should be and are understood as acting in the following way:

$$\mathcal{H}_{-1} \xleftarrow{R_{\lambda\mp i0}(H_r)} \mathcal{H}_1 \xleftarrow{\;V\;} \mathcal{H}_{-1} \xleftarrow{\frac{1}{\pi} \operatorname{Im} R_{\lambda+i0}(H_0)} \mathcal{H}_1.$$

Lemma 6.6. *Let $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$ and let $f \in \mathcal{H}_1$. If $\mathcal{E}_\lambda(H_0)f = 0$, then $\mathfrak{a}_\pm(\lambda; H_r, H_0)f = 0$.*

Proof. This follows from (see 3.14(vii) and (25))

$$\mathcal{E}_\lambda^\diamondsuit(H_0)\mathcal{E}_\lambda(H_0) = \frac{1}{\pi} \operatorname{Im} R_{\lambda+i0}(H_0) \tag{62}$$

(as equality in $\mathcal{L}_1(\mathcal{H}_1, \mathcal{H}_{-1})$) and Proposition 6.5. □

6.2. **Definition of the wave matrix $w_\pm(\lambda; H_r, H_0)$.** Since from now on we need direct integral representations (43) for different operators $H_r = H_0 + V_r$, we denote the fiber Hilbert space, corresponding to H_r by $\mathfrak{h}_\lambda^{(r)}$ or by $\mathfrak{h}_\lambda(H_r)$.

In this section we define the wave matrix $w_\pm(\lambda; H_r, H_0)$ as a form and prove that it is well-defined and bounded, so that it defines an operator.

Definition 6.7. *Let $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$. The wave matrix $w_\pm(\lambda; H_r, H_0)$ is a densely defined form*

$$w_\pm(\lambda; H_r, H_0) \colon \mathfrak{h}_\lambda^{(r)} \times \mathfrak{h}_\lambda^{(0)} \to \mathbb{C},$$

defined by the formula

$$w_\pm(\lambda; H_r, H_0)\left(\mathcal{E}_\lambda(H_r)f, \mathcal{E}_\lambda(H_0)g\right) = \langle f, \mathfrak{a}_\pm(\lambda; H_r, H_0)g\rangle_{1,-1}, \tag{63}$$

where $f, g \in \mathcal{H}_1$.

It is worth to note that this definition depends on endpoint operators H_0 and H_r, but it does not depend on the path $\{H_r\}$ connecting the endpoints.

One needs to show that the wave matrix is well-defined.

Proposition 6.8. *For any $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$ the form $w_\pm(\lambda; H_r, H_0)$ is well-defined, and it is bounded with norm $\leqslant 1$.*

Proof. That $w_\pm(\lambda; H_r, H_0)$ is well-defined follows from Lemma 6.6.

Further, by Schwarz inequality, for any $f, g \in \mathcal{H}_1$,

$$\begin{aligned}
&\frac{y}{\pi} \left|\langle f, R_{\lambda-iy}(H_r) R_{\lambda+iy}(H_0) g\rangle\right| \\
&\qquad = \frac{y}{\pi} \left|\langle R_{\lambda+iy}(H_r) f, R_{\lambda+iy}(H_0) g\rangle\right| \\
&\qquad \leqslant \frac{y}{\pi} \left|\langle R_{\lambda+iy}(H_r) f, R_{\lambda+iy}(H_r) f\rangle\right|^{1/2} \left|\langle R_{\lambda+iy}(H_0) g, R_{\lambda+iy}(H_0) g\rangle\right|^{1/2} \\
&\qquad = \frac{1}{\pi} \left|\langle f, \operatorname{Im} R_{\lambda+iy}(H_r) f\rangle\right|^{1/2} \cdot \left|\langle g, \operatorname{Im} R_{\lambda+iy}(H_0) g\rangle\right|^{1/2}.
\end{aligned} \tag{64}$$

Taking the limit $y \to 0^+$, one gets, using Lemma 6.2, Proposition 6.3 and (62),

$$\left|\langle f, \mathfrak{a}_\pm(\lambda; H_r, H_0) g\rangle_{1,-1}\right| \leqslant \|\mathcal{E}_\lambda(H_r) f\|_{\mathfrak{h}_\lambda^{(r)}} \cdot \|\mathcal{E}_\lambda(H_0) g\|_{\mathfrak{h}_\lambda^{(0)}}.$$

It follows that the wave matrix is bounded with bound less or equal to 1. $\square$

So, the form $w_\pm(\lambda; H_r, H_0)$ is defined on $\mathfrak{h}_\lambda^{(r)} \times \mathfrak{h}_\lambda^{(0)}$. We will identify the form $w_\pm(\lambda)$ with the corresponding operator from $\mathfrak{h}_\lambda^{(0)}$ to $\mathfrak{h}_\lambda^{(r)}$, so that

$$w_\pm(\lambda; H_r, H_0)(\mathcal{E}_\lambda(H_r) f, \mathcal{E}_\lambda(H_0) g) = \langle \mathcal{E}_\lambda(H_r) f, w_\pm(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) g\rangle,$$

where $f, g \in \mathcal{H}_1$. Note that it follows from the definition of $w_\pm(\lambda; H_r, H_0)$ that

$$\mathcal{E}_\lambda^\diamondsuit(H_r) w_\pm(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) = \mathfrak{a}_\pm(\lambda; H_r, H_0). \tag{65}$$

The following proposition follows immediately from the definition of $w_\pm(\lambda; H_r, H_0)$.

Proposition 6.9. *1. Let $\lambda \in \Lambda(H_0; F)$. Then*

$$w_\pm(\lambda; H_0, H_0) = 1.$$

2. Let $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$. Then

$$w_\pm^*(\lambda; H_r, H_0) = w_\pm(\lambda; H_0, H_r). \tag{66}$$

Proof. 1. For any $f, g \in \mathcal{H}_1$, one has

$$\begin{aligned}
\left\langle \mathcal{E}_\lambda(H_0)f, w_\pm(\lambda; H_0, H_0)\mathcal{E}_\lambda(H_0)g \right\rangle_{\mathfrak{h}_\lambda^{(0)}} &= \langle f, \mathfrak{a}_\pm(\lambda; H_0, H_0)g\rangle_{1,-1} && \text{by (63)} \\
&= \frac{1}{\pi} \langle f, \operatorname{Im} R_{\lambda+i0}(H_0)g\rangle_{1,-1} && \text{by (61)} \\
&= \left\langle f, \mathcal{E}_\lambda^\diamondsuit(H_0)\mathcal{E}_\lambda(H_0)g \right\rangle_{1,-1} && \text{by (62)} \\
&= \left\langle \mathcal{E}_\lambda(H_0)f, \mathcal{E}_\lambda(H_0)g \right\rangle_{\mathfrak{h}_\lambda^{(0)}}, && \text{by (25)}
\end{aligned}$$

where (62) has been used. Since $\mathcal{E}_\lambda \mathcal{H}_1$ is, by definition, dense in $\mathfrak{h}_\lambda$ (see (41)) and since, by Proposition 6.8, the wave matrix $w_\pm(\lambda; H_r, H_0)$ is bounded, it follows from the last equality that $w_\pm(\lambda; H_0, H_0) = 1$.

2. This follows directly from the definition of $w_\pm(\lambda; H_r, H_0)$. The details are omitted since later we derive this property of the wave matrix from the multiplicative property. □

6.3. **Multiplicative property of the wave matrix.** We have shown that the wave matrix is a bounded operator from $\mathfrak{h}_\lambda^{(0)}$ to $\mathfrak{h}_\lambda^{(r)}$. The next thing to do is to show that it is a unitary operator. Unitary property of the wave matrix is a consequence of the multiplicative property and the norm bound $\|w_\pm\| \leqslant 1$.

In this subsection we establish the multiplicative property of the wave matrix. We shall intensively use objects such as $\varphi_j(\lambda+iy)$, $b_j(\lambda+iy)$ and so on, associated to a self-adjoint operator H_r on a fixed framed Hilbert space $(\mathcal{H}, F)$. Which self-adjoint operator these objects are associated with will be clear from the context. For example, if one meets an expression $R_{\lambda+iy}(H_r)b_j(\lambda+iy)$, then this means that $b_j(\lambda+iy)$ is associated with H_r.

Lemma 6.10. *Let $\lambda \in \Lambda(H_0; F)$. If $f = \sum_{k=1}^\infty \beta_k \kappa_k \varphi_k \in \mathcal{H}_1$ (so that $(\beta_j) \in \ell_2$), then*

$$\left\langle \mathcal{E}_{\lambda+iy}(H_0)f, e_j(\lambda+iy) \right\rangle_{\ell_2} = \alpha_j(\lambda+iy) \left\langle \beta, e_j(\lambda+iy) \right\rangle_{\ell_2}.$$

Proof. One has

$$
\begin{aligned}
\langle \mathcal{E}_{\lambda+iy}(H_0)f, e_j(\lambda+iy)\rangle &= \left\langle \mathcal{E}_{\lambda+iy}(H_0)\sum_{k=1}^{\infty}\beta_k\kappa_k\varphi_k, e_j(\lambda+iy)\right\rangle \\
&= \left\langle \sum_{k=1}^{\infty}\beta_k\kappa_k\mathcal{E}_{\lambda+iy}(H_0)\varphi_k, e_j(\lambda+iy)\right\rangle \\
&= \left\langle \sum_{k=1}^{\infty}\beta_k\eta_k(\lambda+iy), e_j(\lambda+iy)\right\rangle && \text{by (31)} \\
&= \sum_{k=1}^{\infty}\bar\beta_k\left\langle \eta_k(\lambda+iy), e_j(\lambda+iy)\right\rangle \\
&= \sum_{k=1}^{\infty}\bar\beta_k\alpha_j(\lambda+iy)e_{kj}(\lambda+iy) \\
&= \alpha_j(\lambda+iy)\left\langle \beta, e_j(\lambda+iy)\right\rangle_{\ell_2}.
\end{aligned}
$$

The second equality holds, since $\mathcal{E}_{\lambda+iy}$ is a bounded operator from $\mathcal{H}_1$ to ℓ_2. The fourth equality holds, since the series $\sum\limits_{k=1}^{\infty}\beta_k\eta_k$ is absolutely convergent. The fifth equality holds, since $e_j(\lambda+iy)$ is an eigenvector of the matrix $\eta(\lambda+iy)$ with the eigenvalue $\alpha_j(\lambda+iy)$. $\square$

Lemma 6.11. *Let $\lambda \in \Lambda(H_0;F)$ and $f \in \mathcal{H}_1$. If j is an index of zero-type, then*

$$\langle \mathcal{E}_{\lambda+iy}(H_0)f, e_j(\lambda+iy)\rangle \to 0,$$

as $y \to 0$.

Proof. Using Lemma 6.10 (and its representation for f) and the definition of $e_j(\lambda+iy)$ we have

$$
\begin{aligned}
|\langle \mathcal{E}_{\lambda+iy}(H_0)f, e_j(\lambda+iy)\rangle| &= \alpha_j(\lambda+iy)\left|\langle \beta, e_j(\lambda+iy)\rangle_{\ell_2}\right| \\
&\leqslant \alpha_j(\lambda+iy)\|\beta\|\,\|e_j(\lambda+iy)\| = \alpha_j(\lambda+iy)\|\beta\|.
\end{aligned}
$$

If j is an index of zero type (see subsection 3.9) then, by definition, $\alpha_j(\lambda+iy) \to 0$ as $y \to 0$. The proof is complete. $\square$

Lemma 6.12. *Let $\lambda \in \Lambda(H_r;F) \cap \Lambda(H_0;F)$. If j is of zero-type, then for any $f \in \mathcal{H}_1$,*

$$\frac{y}{\pi}\langle R_{\lambda\pm iy}(H_r)f, R_{\lambda\pm iy}(H_0)b_j(\lambda+iy)\rangle \to 0, \tag{67}$$

as $y \to 0$.

Proof. The first equality in (60) and 3.14(i) imply that

$$
\begin{aligned}
&\frac{y}{\pi}\langle R_{\lambda\pm iy}(H_r)f, R_{\lambda\pm iy}(H_0)b_j(\lambda+iy)\rangle \\
&\qquad = \langle \mathcal{E}_{\lambda+iy}(H_0)[1+V_rR_{\lambda\pm iy}(H_r)]f, \mathcal{E}_{\lambda+iy}(H_0)b_j(\lambda+iy)\rangle \\
&\qquad = \langle \mathcal{E}_{\lambda+iy}(H_0)[1+V_rR_{\lambda\pm iy}(H_r)]f, e_j(\lambda+iy)\rangle,
\end{aligned} \tag{68}
$$

where the second equality follows from the definition (36) of $b_j(\lambda+iy)$. Since by Lemma 6.1 the resolvent $R_{\lambda\pm iy}(H_r)$ converges as an operator from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$, and since V maps $\mathcal{H}_{-1}$ to $\mathcal{H}_1$ (see (50)), it follows that the vector $VR_{\lambda\pm iy}(H_r)f$ converges in $\mathcal{H}_1$ as $y \to 0$. Now, applying Lemma 6.10 and using the fact that for indices of zero type j the eigenvalues $\alpha_j(\lambda + iy)$ converge to 0, we conclude that the expression in (67) converges to 0 as $y \to 0$. $\square$

Lemma 6.13. *If a non-increasing sequence $f_1, f_2, \dots$ of continuous functions on $[0,1]$ converges pointwise to 0, then it also converges to 0 uniformly.*

Proof. Let $\varepsilon > 0$ and let $x \in [0,1]$. Since $f_n(x) \to 0$, there exists $N(x) \in \mathbb{N}$, such that for all $n \geqslant N(x) \quad f_n(x) < \varepsilon/2$. Let U_x be a neighbourhood of x such that $f_{N(x)}(y) < \varepsilon$ for all $y \in U_x$. Then for all $n \geqslant N(x)$ and for all $y \in U_x \quad f_n(y) < \varepsilon$. If we choose a finite cover $U_{x_1}, \dots, U_{x_m}$ of $[0,1]$, and let $N = \max\{N(x_j)\}$, then for any $x \in [0,1]$ and any $n \geqslant N$ we have $f_n(x) < \varepsilon$. $\square$

Lemma 6.14. *Let $\lambda \in \Lambda(H_0; F)$. The sum*

$$\sum_{j=N}^{\infty} \alpha_j(\lambda + iy)^2$$

converges to 0 as $N \to \infty$ uniformly with respect to $y \in [0,1]$.

Proof. Let $f_N(y)$ be this sum. Since $f_1(y) = \|\eta(\lambda + iy)\|_2^2$, it follows from 3.7(iii) and (iv), that $f_1(y)$, and, consequently, all $f_N(y)$ are continuous functions of y in $[0,1]$. So, we have a non-increasing sequence $f_N(y)$ of continuous non-negative functions, converging pointwise to 0 as $N \to \infty$. It follows from Lemma 6.13 that the sequence $f_N(y)$ converges to 0 as $N \to \infty$ uniformly with respect to $y \in [0,1]$. $\square$

Lemma 6.15. *Let $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$. If $f \in \mathcal{H}_1$, then the sequence*

$$\left(\frac{y}{\pi}\right)^2 \sum_{j=N}^{\infty} \left|\langle R_{\lambda\pm iy}(H_r)f, R_{\lambda\pm iy}(H_0)b_j(\lambda+iy)\rangle\right|^2, \quad N = 1, 2, \dots$$

converges to 0 as $N \to \infty$, uniformly with respect to $y > 0$.

Proof. We prove the lemma for the plus sign. The formula (68) and Lemma 6.10 imply that

$$\begin{aligned}(E) &:= \left(\frac{y}{\pi}\right)^2 \sum_{j=N}^{\infty} \left|\langle R_{\lambda+iy}(H_r)f, R_{\lambda+iy}(H_0)b_j(\lambda+iy)\rangle\right|^2 \\ &= \sum_{j=N}^{\infty} \left|\langle \mathcal{E}_{\lambda+iy}(H_0)[1 + V_r R_{\lambda+iy}(H_r)]f, e_j(\lambda+iy)\rangle\right|^2 \\ &= \sum_{j=N}^{\infty} \left|\alpha_j(\lambda+iy)\langle \beta(\lambda+iy), e_j(\lambda+iy)\rangle\right|^2,\end{aligned}$$

where $\beta(\lambda+iy) = (\beta_k(\lambda+iy)) \in \ell_2$, and

$$[1 + V_r R_{\lambda+iy}(H_r)]f = \sum_{k=1}^{\infty} \beta_k(\lambda+iy)\kappa_k\varphi_k \in \mathcal{H}_1.$$

Since $[1 + V_r R_{\lambda+iy}(H_r)]f$ converges in $\mathcal{H}_1$ as $y \to 0$, the sequence $(\beta_k(\lambda+iy))$ converges in ℓ_2 as $y \to 0$. It follows that $\|\beta(\lambda+iy)\|_{\ell_2} \leqslant C$ for all $y \in [0,1]$. Hence,

$$(E) \leqslant C^2 \sum_{j=N}^{\infty} \alpha_j(\lambda+iy)^2.$$

By Lemma 6.14, the last expression converges to 0 uniformly. □

In the following theorem we prove the multiplicative property of the wave matrix. This is a well-known property [Y], but the novelty is that we give an explicit set of full measure, such that for all λ from that set the wave matrices are explicitly defined and the multiplicative property holds.

Theorem 6.16. *Let $\{H_r\}$ be a path satisfying Assumption 5.1. If $\lambda \in \Lambda(H_0;F)$ and if r_0, r_1, r_2 are not resonance points of the path $\{H_r\}$ for this λ (that is, if $r_0, r_1, r_2 \notin R(\lambda;\{H_r\},F)$), then*

$$w_\pm(\lambda; H_{r_2}, H_{r_0}) = w_\pm(\lambda; H_{r_2}, H_{r_1}) w_\pm(\lambda; H_{r_1}, H_{r_0}).$$

Proof. We prove this equality for + sign. Let $f, g \in \mathcal{H}_1$. It follows from 3.15(vi) that

$$\begin{aligned} &\frac{y}{\pi}\langle R_{\lambda+iy}(H_{r_2})f, R_{\lambda+iy}(H_{r_0})g\rangle \\ &\qquad = \left(\frac{y}{\pi}\right)^2 \sum_{j=1}^{\infty} \langle R_{\lambda+iy}(H_{r_2})f, R_{\lambda+iy}(H_{r_1})b_j(\lambda+iy)\rangle \\ &\qquad\qquad \cdot \langle R_{\lambda+iy}(H_{r_1})b_j(\lambda+iy), R_{\lambda+iy}(H_{r_0})g\rangle, \end{aligned} \tag{69}$$

where the series converges absolutely, since the set of vectors $\left\{\sqrt{\frac{y}{\pi}} R_{\lambda+iy}(H_{r_1})b_j(\lambda+iy)\right\}$ is orthonormal and complete (see 3.15(vi)). Applying Schwarz inequality to (69) and using Lemma 6.15, by Vitali's Theorem 2.1, one can take the limit $y \to 0$ in this formula. By Lemma 6.12, the summands with zero-type j disappear after taking the limit $y \to 0$.

So, it follows from the Definition 6.4 of $\mathfrak{a}_\pm$, that

$$\begin{aligned} &\langle f, \mathfrak{a}_+(\lambda; H_{r_2}, H_{r_0})g\rangle_{1,-1} \\ &\quad = \sum_{j=1}^{\infty} \langle f, \mathfrak{a}_+(\lambda; H_{r_2}, H_{r_1})b_j(\lambda+i0)\rangle_{1,-1} \langle b_j(\lambda+i0), \mathfrak{a}_+(\lambda; H_{r_1}, H_{r_0})g\rangle_{1,-1}, \end{aligned} \tag{70}$$

where the summation is over indices of non-zero type. By definition (63) of $w_\pm$, it follows from (70) that

(71)
$$\begin{aligned}
&\langle \mathcal{E}_\lambda^{(r_2)} f, w_\pm(\lambda; H_{r_2}, H_{r_0}) \mathcal{E}_\lambda^{(r_0)} g \rangle \\
&\quad = \sum_{j=1}^\infty \langle \mathcal{E}_\lambda^{(r_2)} f, w_\pm(\lambda; H_{r_2}, H_{r_1}) \mathcal{E}_\lambda^{(r_1)} b_j(\lambda + i0) \rangle \, \langle \mathcal{E}_\lambda^{(r_1)} b_j(\lambda + i0), w_\pm(\lambda; H_{r_1}, H_{r_0}) \mathcal{E}_\lambda^{(r_0)} g \rangle \\
&\quad = \sum_{j=1}^\infty \langle \mathcal{E}_\lambda^{(r_2)} f, w_\pm(\lambda; H_{r_2}, H_{r_1}) e_j(\lambda + i0) \rangle \, \langle e_j(\lambda + i0), w_\pm(\lambda; H_{r_1}, H_{r_0}) \mathcal{E}_\lambda^{(r_0)} g \rangle \\
&\quad = \langle \mathcal{E}_\lambda^{(r_2)} f, w_\pm(\lambda; H_{r_2}, H_{r_1}) w_\pm(\lambda; H_{r_1}, H_{r_0}) \mathcal{E}_\lambda^{(r_0)} g \rangle,
\end{aligned}$$

where in the last equality Lemma 4.6 was used. Since the set $\mathcal{E}_\lambda \mathcal{H}_1$ is dense in $\mathfrak{h}_\lambda$, the proof is complete. □

Corollary 6.17. *Let $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$. Then $w_\pm(\lambda; H_r, H_0)$ is a unitary operator from $\mathfrak{h}_\lambda^{(0)}$ to $\mathfrak{h}_\lambda^{(r)}$ and (66) holds.*

Proof. Indeed, using the first part of Proposition 6.9 and the multiplicative property of the wave matrix (Theorem 6.16), one infers that

$$w_\pm(\lambda; H_0, H_r) w_\pm(\lambda; H_r, H_0) = w_\pm(\lambda; H_0, H_0) = 1$$

and

$$w_\pm(\lambda; H_r, H_0) w_\pm(\lambda; H_0, H_r) = w_\pm(\lambda; H_r, H_r) = 1.$$

Since by Proposition 6.8 $\|w_\pm(\lambda; H_r, H_0)\| \leqslant 1$, it follows that $w_\pm(\lambda; H_r, H_0)$ is a unitary operator and

$$w_\pm^*(\lambda; H_r, H_0) = w_\pm(\lambda; H_0, H_r).$$

□

Remark 6.18. There is an essential difference between $\sqrt{\frac{y}{\pi}} R_{\lambda+iy}(H_0)$ (or $\sqrt{\frac{y}{\pi}} R_{\lambda-iy}(H_0)$) and $\mathcal{E}_{\lambda+iy}(H_0)$. While they have some common features (see formulae 3.15(iv) and 3.15(v)), the second operator is better than the first one. Actually, as it can be seen from the definitions of $\sqrt{\frac{y}{\pi}} R_{\lambda+iy}(H_0)$ and $\mathcal{E}_{\lambda+iy}(H_0)$, these operators "differ" by the phase part. This statement is enforced by the fact that the $\mathcal{L}_2(\mathcal{H}_1, \mathcal{H})$ norm of the difference

$$\sqrt{\frac{y}{\pi}} R_{\lambda+iy}(H_0) - \sqrt{\frac{y_1}{\pi}} R_{\lambda+iy_1}(H_0)$$

remains bounded as $y, y_1 \to 0$, even though it does not converge to 0. Convergence is hindered by the non-convergent phase part, which is absent in $\mathcal{E}_{\lambda+iy}(H_0)$.

6.4. **The wave operator.** Recall that a family of operators $A_\lambda \colon \mathfrak{h}_\lambda(H_0) \to \mathfrak{h}_\lambda(H_1)$ is measurable, if it maps measurable vector-functions to measurable vector-functions. Recall that if

$$A = \int_\Lambda^\oplus A(\lambda)\, d\lambda \quad \text{and} \quad B = \int_\Lambda^\oplus B(\lambda)\, d\lambda,$$

then

$$AB = \int_{\Lambda}^{\oplus} A(\lambda)B(\lambda)\,d\lambda.$$

We define the wave operator $W_\pm$ as the direct integral of wave matrices:

$$W_\pm(H_r, H_0) := \int_{\Lambda(H_r;F)\cap\Lambda(H_0;F)}^{\oplus} w_\pm(\lambda; H_r, H_0)\,d\lambda. \tag{72}$$

It is clear from (63) that the operator-function

$$\Lambda(H_r; F) \cap \Lambda(H_0; F) \ni \lambda \mapsto w_\pm(\lambda; H_r, H_0)$$

is measurable, so that the integral above makes sense.

The following well-known theorem (cf. [Y, Chapter 2]) is a direct consequence of the definition (72) of the wave operator $W_\pm$, Theorem 6.16 and Corollary 6.17.

Theorem 6.19. *Let $\{H_r\}$ be a path of self-adjoint operators which satisfies Assumption 5.1. The wave operator $W_\pm(H_r, H_0)\colon \mathcal{H}^{(a)}(H_0) \to \mathcal{H}^{(a)}(H_r)$ possesses the following properties.*

(i) $W_\pm(H_r, H_0)$ *is a unitary operator.*
(ii) $W_\pm(H_r, H_0) = W_\pm(H_r, H_s)W_\pm(H_s, H_0)$.
(iii) $W_\pm^*(H_r, H_0) = W_\pm(H_0, H_r)$.
(iv) $W_\pm(H_0, H_0) = 1$.

If we define $W_\pm(H_r, H_0)$ to be zero on the singular subspace $\mathcal{H}^{(s)}(H_0)$, then the part (iv) becomes

$$W_\pm(H_0, H_0) = P^{(a)}(H_0).$$

That is, $W_\pm(H_r, H_0)$ becomes a partial isometry with initial space $\mathcal{H}^{(a)}(H_0)$ and final space $\mathcal{H}^{(a)}(H_r)$. So,

$$W_\pm(H_r, H_0) = W_\pm(H_r, H_0)P^{(a)}(H_0) = P^{(a)}(H_r)W_\pm(H_r, H_0).$$

Theorem 6.20. *(cf. [Y, Theorem 2.1.4]) For any bounded measurable function h on $\mathbb{R}$*

$$h(H_r)W_\pm(H_r, H_0) = W_\pm(H_r, H_0)h(H_0). \tag{73}$$

Also,

$$H_r W_\pm(H_r, H_0) = W_\pm(H_r, H_0)H_0. \tag{74}$$

Proof. This follows from the definition (72) of $W_\pm$ and Theorem 4.18. □

As a consequence, we also get the Kato-Rosenblum theorem.

Corollary 6.21. *The operators $H_0^{(a)}$ and $H_1^{(a)}$, considered as operators on absolutely continuous subspaces $\mathcal{H}^{(a)}(H_0)$ and $\mathcal{H}^{(a)}(H_1)$ respectively, are unitarily equivalent.*

This follows from (74).

7. Connection with time-dependent definition of the wave operator

In this section we show that the wave operator defined by (72) coincides with the classical time-dependent definition. In this subsection I follow [Y]. Though the proofs follow almost verbatim those in [Y] (in [Y] the proofs are given in a more general setting), they are given here for reader's convenience and completeness. On the other hand, availability of the evaluation operator $\mathcal{E}_\lambda$ allows to simplify the proofs slightly.

In abstract scattering theory the wave operator is usually defined by the formula (cf. e.g. [Y, (2.1.1)])

$$W_\pm(H_r, H_0) = \lim_{t\to\pm\infty} e^{itH_r} e^{-itH_0} P^{(a)}(H_0) =: \overset{\mathrm{s}}{W}_\pm(H_r, H_0), \tag{75}$$

where the limit is taken in the strong operator topology. Since we define the wave operator in a different way, this formula becomes a theorem.

We denote by $P_r^{(a)}$ the projection $P^{(a)}(H_r)$.

The weak wave operators $\overset{\mathrm{w}}{W}_\pm$ are defined, if they exist, by the formula

$$\overset{\mathrm{w}}{W}_\pm(H_r, H_0) := \lim_{t\to\pm\infty} P_r^{(a)} e^{itH_r} e^{-itH_0} P_0^{(a)}, \tag{76}$$

where the limit is taken in the weak operator topology.

Proof of the existence of the wave operator in the strong operator topology uses the existence of the weak wave operator and the multiplicative property of it. The proof of the latter constitutes the main difficulty of the stationary approach.

The following lemma is taken from [Y, Lemma 5.3.1].

Lemma 7.1. *If $g \in \mathcal{H}$ is such that $\|\mathcal{E}_\lambda g\|_{\mathfrak{h}_\lambda} \leqslant N$ for a.e. $\lambda \in \Lambda(H_0; F)$, then*

$$\int_{-\infty}^{\infty} \left\| F e^{-itH_0} P_0^{(a)} g \right\|^2 dt \leqslant 2\pi N^2 \|F\|_2^2 .$$

Proof. (A) For any frame vector φ_j the following estimate holds:

$$\int_{-\infty}^{\infty} \left| \left\langle e^{-itH_0} P_0^{(a)} g, \varphi_j \right\rangle \right|^2 dt \leqslant 2\pi N^2.$$

Proof. Note that $g(\lambda)$ is defined for a.e. $\lambda \in \Lambda(H_0; F)$ as an element of the direct integral $\mathcal{H}$. It follows from Theorem 4.18 and Lemma 4.12 that

$$\begin{aligned}\left\langle e^{-itH_0} P_0^{(a)} g, \varphi_j \right\rangle &= \int_{\Lambda(H_0;F)} e^{-i\lambda t} \left\langle g(\lambda), \varphi_j(\lambda) \right\rangle d\lambda \\ &= \sqrt{2\pi} \hat{f}_j(t),\end{aligned}$$

where $f_j(\lambda) = \langle g(\lambda), \varphi_j(\lambda)\rangle$ and $\hat f_j$ is the Fourier transform of f_j. It follows that

$$\begin{aligned}(E) &:= \int_{-\infty}^{\infty} \left|\left\langle e^{-itH_0}P_0^{(a)}g, \varphi_j\right\rangle\right|^2\,dt \\ &= 2\pi\int_{-\infty}^{\infty}\left|\hat f_j(t)\right|^2\,dt = 2\pi\int_{\Lambda(H_0;F)} |f_j(\lambda)|^2\,d\lambda \\ &= 2\pi\int_{\Lambda(H_0;F)} |\langle g(\lambda),\varphi_j(\lambda)\rangle|^2\,d\lambda \leqslant 2\pi N^2\int_{\Lambda(H_0;F)}\|\varphi_j(\lambda)\|^2\,d\lambda \\ &\leqslant 2\pi N^2.\end{aligned}$$

We write here $\Lambda(H_0;F)$ instead of $\mathbb{R}$, but since $\Lambda(H_0;F)$ has full Lebesgue measure, it makes no difference. The proof of (A) is complete.

(B) Using the Parseval equality one has (recall that (ψ_j) is the orthonormal basis from the definition (20) of the frame operator F)

$$\begin{aligned}(E) := \int_{-\infty}^{\infty}\left\|Fe^{-itH_0}P_0^{(a)}g\right\|^2\,dt &= \int_{-\infty}^{\infty}\sum_{j=1}^{\infty}\left|\left\langle Fe^{-itH_0}P_0^{(a)}g, \psi_j\right\rangle\right|^2\,dt \\ &= \int_{-\infty}^{\infty}\sum_{j=1}^{\infty}\kappa_j^2\left|\left\langle e^{-itH_0}P_0^{(a)}g, \varphi_j\right\rangle\right|^2\,dt \\ &= \sum_{j=1}^{\infty}\kappa_j^2\int_{-\infty}^{\infty}\left|\left\langle e^{-itH_0}P_0^{(a)}g, \varphi_j\right\rangle\right|^2\,dt.\end{aligned}$$

Now, it follows from (A) that

$$(E) \leqslant \sum_{j=1}^{\infty}\kappa_j^2\cdot 2\pi N^2 = 2\pi N^2\,\|F\|_2^2\,.$$

The proof is complete. ☐

For the following theorem, see e.g. [Y, Theorem 5.3.2]

Theorem 7.2. *The weak wave operators (76) exist.*

Proof. (A) For any $f, f_0 \in \mathcal{H}$ the estimate

$$\begin{aligned}&\left|\left\langle e^{-it_2H_r}e^{-it_2H_0}f_0, f\right\rangle - \left\langle e^{-it_1H_r}e^{-it_1H_0}f_0, f\right\rangle\right| \\ &\qquad\leqslant \|J_r\|\left(\int_{t_1}^{t_2}\left\|Fe^{-it_2H_0}f_0\right\|^2\,dt\right)^{1/2}\left(\int_{t_1}^{t_2}\left\|Fe^{-it_2H_r}f\right\|^2\,dt\right)^{1/2}.\end{aligned}$$

holds.

Proof. For any $f, f_0 \in \mathcal{H}$,

$$\begin{aligned}\frac{d}{dt}\left\langle e^{-itH_0}f_0, e^{-itH_r}f\right\rangle &= \left\langle (-iH_0)e^{-itH_0}f_0, e^{-itH_r}f\right\rangle + \left\langle e^{-itH_0}f_0, (-iH_r)e^{-itH_r}f\right\rangle \\ &= -i\left\langle (H_r - H_0)e^{-itH_0}f_0, e^{-itH_r}f\right\rangle \\ &= -i\left\langle V_r e^{-itH_0}f_0, e^{-itH_r}f\right\rangle \\ &= -i\left\langle J_r F e^{-itH_0}f_0, F e^{-itH_r}f\right\rangle,\end{aligned}$$

where in the last equality the decomposition $V_r = F^* J_r F$ was used. It follows that

$$\begin{aligned}&\left\langle e^{-it_2H_r}e^{-it_2H_0}f_0, f\right\rangle - \left\langle e^{-it_1H_r}e^{-it_1H_0}f_0, f\right\rangle \\ &\qquad = -i\int_{t_1}^{t_2}\left\langle J_r F e^{-it_2H_0}f_0, F e^{-it_2H_r}f\right\rangle\, dt.\end{aligned}$$

Using the Schwarz inequality, this implies that

$$\begin{aligned}&\left|\left\langle e^{-it_2H_r}e^{-it_2H_0}f_0, f\right\rangle - \left\langle e^{-it_1H_r}e^{-it_1H_0}f_0, f\right\rangle\right| \\ &\qquad \leqslant \|J_r\|\int_{t_1}^{t_2}\left\|F e^{-it_2H_0}f_0\right\|\left\|F e^{-it_2H_r}f\right\|\, dt \\ &\qquad \leqslant \|J_r\|\left(\int_{t_1}^{t_2}\left\|F e^{-it_2H_0}f_0\right\|^2 dt\right)^{1/2}\left(\int_{t_1}^{t_2}\left\|F e^{-it_2H_r}f\right\|^2 dt\right)^{1/2}.\end{aligned}$$

(B) Let $N \in \mathbb{R}$. Let $g, g_0 \in \mathcal{H}$ be such that for a.e. $\lambda \in \Lambda(H_0; F)$

$$\left\|\mathcal{E}_\lambda(H_0)P_0^{(a)}g_0\right\|_{\mathfrak{h}_\lambda^{(0)}} \leqslant N \text{ and } \left\|\mathcal{E}_\lambda(H_r)P_r^{(a)}g\right\|_{\mathfrak{h}_\lambda^{(r)}} \leqslant N. \tag{77}$$

Applying the estimate (A) to the pair of vectors $f = P^{(a)}(H_r)g$ and $f_0 = P^{(a)}(H_0)g_0$, it now follows from the estimates (77) and Lemma 7.1, that

$$\begin{aligned}&\left|\left\langle e^{-it_2H_r}e^{-it_2H_0}P^{(a)}(H_0)g_0, P^{(a)}(H_r)g\right\rangle - \left\langle e^{-it_1H_r}e^{-it_1H_0}P^{(a)}(H_0)g_0, P^{(a)}(H_r)g\right\rangle\right| \\ &\qquad\qquad \leqslant \|J_r\|\cdot 2\pi N^2\,\|F\|_2^2.\end{aligned}$$

Consequently, the right hand side vanishes, when $t_1, t_2 \to \pm\infty$. It follows that the limits

$$\lim_{t\to\pm\infty}\left\langle P_r^{(a)}e^{-itH_r}e^{-itH_0}P_0^{(a)}g_0, g\right\rangle$$

exist. Since the set of vectors g_0, g, which satisfy the estimate (77) for some N, is dense in $\mathcal{H}$, it follows from the last estimate that the weak wave operators (76) exist. □

The following theorem and its proof follow verbatim [Y, Theorem 2.2.1]

Theorem 7.3. *If the weak wave operators* $\overset{\mathrm{w}}{W}_\pm(H_r, H_0)$ *exist and*

$$\overset{\mathrm{w}}{W}_\pm(H_r, H_0)^*\,\overset{\mathrm{w}}{W}_\pm(H_r, H_0) = P_0^{(a)}, \tag{78}$$

then the strong wave operators $\overset{\mathrm{s}}{W}_\pm(H_r, H_0)$ *exist and coincide with the weak wave operators* $\overset{\mathrm{w}}{W}_\pm(H_r, H_0)$.

Proof. We have

$$\begin{aligned} E_\pm(t) &:= \left\| e^{itH_r}e^{-itH_0}P_0^{(a)}f - \overset{\mathrm{w}}{W}_\pm f\right\|^2 \\ &= \left\langle e^{itH_r}e^{-itH_0}P_0^{(a)}f - \overset{\mathrm{w}}{W}_\pm f, e^{itH_r}e^{-itH_0}P_0^{(a)}f - \overset{\mathrm{w}}{W}_\pm f\right\rangle \\ &= \left\langle P_0^{(a)}f, f\right\rangle - 2\,\mathrm{Re}\left\langle e^{itH_r}e^{-itH_0}P_0^{(a)}f, \overset{\mathrm{w}}{W}_\pm f\right\rangle + \left\langle \overset{\mathrm{w}}{W}_\pm f, \overset{\mathrm{w}}{W}_\pm f\right\rangle. \end{aligned}$$

Since $\overset{\mathrm{w}}{W}_\pm = P_r^{(a)}\ \overset{\mathrm{w}}{W}_\pm$, it follows from (76) that the second term on the right-hand side of this equality converges to $-2\left\langle \overset{\mathrm{w}}{W}_\pm f, \overset{\mathrm{w}}{W}_\pm f\right\rangle$ as $t \to \pm\infty$. It follows from this and (78) that

$$\lim_{t\to\pm\infty} E_\pm(t) = \left\langle P_0^{(a)}f, f\right\rangle - \left\langle \overset{\mathrm{w}}{W}{}^*_\pm\ \overset{\mathrm{w}}{W}_\pm f, f\right\rangle = 0.$$

That is, the strong wave operators $\overset{\mathrm{s}}{W}_\pm$ exist and are equal to $\overset{\mathrm{w}}{W}_\pm$. ☐

The next theorem is taken from [Y, Chapter 2].

Theorem 7.4. *The strong wave operators $\overset{\mathrm{s}}{W}_\pm$ exist and coincide with $W_\pm$.*

Proof. (A) Let $f, g \in \mathcal{H}_1$ and let $\Lambda = \Lambda(H_r; F) \cap \Lambda(H_0; F)$. For every $\lambda \in \Lambda$ the vectors $f^{(r)}(\lambda) = \mathcal{E}_\lambda(H_r)f$ and $g^{(0)}(\lambda) = \mathcal{E}_\lambda(H_0)g$ are well-defined and the functions $f^{(r)}(\cdot)$ and $g^{(0)}(\cdot)$ are $\mathcal{H}$-measurable in the corresponding direct integrals, so that

$$\tilde f := P_r^{(a)}f = \int_\Lambda^\oplus f^{(r)}(\lambda)\,d\lambda, \qquad \tilde g := P_0^{(a)}g = \int_\Lambda^\oplus g^{(0)}(\lambda)\,d\lambda.$$

It follows from the definitions (72) and (63) of the wave operator $W_\pm$ and the wave matrix $w_\pm(\lambda)$ that

$$\begin{aligned} \left\langle \tilde f, W_\pm(H_r, H_0)\tilde g\right\rangle &= \int_\Lambda \left\langle f^{(r)}(\lambda), w_\pm(\lambda; H_r, H_0)g^{(0)}(\lambda)\right\rangle_{\mathfrak{h}_\lambda^{(r)}} d\lambda \\ &= \int_\Lambda \left\langle \tilde f, \mathfrak{a}_\pm(\lambda; H_r, H_0)\tilde g\right\rangle_{1,-1} d\lambda. \end{aligned}$$

By definition (6.4) of the operators $\mathfrak{a}_\pm(\lambda)$, it follows from the last equality that

$$\left\langle \tilde f, W_\pm(H_r, H_0)\tilde g\right\rangle = \int_\Lambda \lim_{y\to 0}\frac{y}{\pi}\left\langle R_{\lambda\pm iy}(H_r)\tilde f, R_{\lambda\pm iy}(H_0)\tilde g\right\rangle d\lambda. \tag{79}$$

(B) Claim: the limit and the integral can be interchanged.

Let Y be a Borel subset of Λ and let

$$f_y = \frac{y}{\pi}\left\langle R_{\lambda\pm iy}(H_r)\tilde f, R_{\lambda\pm iy}(H_0)\tilde g\right\rangle.$$

The Schwarz inequality implies

$$\begin{aligned}\int_Y |f_y| \, d\lambda &\leqslant \frac{y}{\pi} \int_Y \left\| R_{\lambda \pm iy}(H_r) \tilde f \right\| \| R_{\lambda \pm iy}(H_0) \tilde g \| \, d\lambda \\ &\leqslant \left(\frac{y}{\pi} \int_Y \left\| R_{\lambda \pm iy}(H_r) \tilde f \right\|^2 \, d\lambda \right)^{1/2} \left(\frac{y}{\pi} \int_Y \| R_{\lambda \pm iy}(H_0) \tilde g \|^2 \, d\lambda \right)^{1/2} \\ &\leqslant \left(\frac{1}{\pi} \int_Y \left\langle \operatorname{Im} R_{\lambda + iy}(H_r) \tilde f, \tilde f \right\rangle \, d\lambda \right)^{1/2} \left(\frac{1}{\pi} \int_Y \langle \operatorname{Im} R_{\lambda + iy}(H_0) \tilde g, \tilde g \rangle \, d\lambda \right)^{1/2}\end{aligned}$$

Since $\tilde f$ is an absolutely continuous vector for H_r and since $\tilde g$ is an absolutely continuous vector for H_0, the functions $\frac{1}{\pi} \left\langle \operatorname{Im} R_{\lambda+iy}(H_r) \tilde f, \tilde f \right\rangle$ and $\frac{1}{\pi} \langle \operatorname{Im} R_{\lambda+iy}(H_0) \tilde g, \tilde g \rangle$ are Poisson integrals of summable functions $\frac{d}{d\lambda} \left\langle E_\lambda^{H_r} \tilde f, \tilde f \right\rangle$ and $\frac{d}{d\lambda} \langle E_\lambda^{H_0} \tilde g, \tilde g \rangle$ respectively. From Lemma 2.2 and from the above estimate it now follows that for f_y the conditions of Vitali's Theorem 2.1 hold. Hence, Vitali's theorem completes the proof of (A).

(C) Claim: $\overset{\mathrm{w}}{W}_\pm(H_r, H_0) = W_\pm(H_r, H_0)$.

Proof. Using [Y, (2.7.2)], it follows from (79) that

$$\left\langle \tilde f, W_\pm(H_r, H_0) \tilde g \right\rangle = \lim_{\varepsilon \to 0} 2\varepsilon \int_0^\infty e^{-2\varepsilon t} \left\langle e^{\mp i t H_r} \tilde f, e^{\mp i t H_0} \tilde g \right\rangle \, dt.$$

Since, by Theorem 7.2, the function $t \mapsto \left\langle e^{\mp i t H_r} \tilde f, e^{\mp i t H_0} \tilde g \right\rangle$ has a limit, as $t \to \infty$, equal to $\left\langle \tilde f, \overset{\mathrm{w}}{W}_\pm(H_r, H_0) \tilde g \right\rangle$, it follows that the right hand side of the last equality is also equal to $\left\langle \tilde f, \overset{\mathrm{w}}{W}_\pm(H_r, H_0) \tilde g \right\rangle$. Hence,

$$\left\langle \tilde f, W_\pm(H_r, H_0) \tilde g \right\rangle = \left\langle \tilde f, \overset{\mathrm{w}}{W}_\pm(H_r, H_0) \tilde g \right\rangle .$$

Since for any self-adjoint operator H the set $P^{(a)}(H)\mathcal{H}_1$ is dense in $\mathcal{H}^{(a)}(H)$ and since both operators $\overset{\mathrm{w}}{W}_\pm(H_r, H_0)$ and $W_\pm(H_r, H_0)$ vanish on singular subspace $\mathcal{H}^{(s)}(H_0)$ of H_0, it follows that $W_\pm(H_r, H_0) = \overset{\mathrm{w}}{W}_\pm(H_r, H_0)$.

(D) Since for $W_\pm$ the multiplicative property holds (Theorem 6.19(ii)), it follows from (C) that the multiplicative property holds also for the weak wave operator $\overset{\mathrm{w}}{W}_\pm$. Further, by Theorem 7.3 existence of the weak wave operator and the multiplicative property imply that the strong wave operator $\overset{\mathrm{s}}{W}_\pm$ exists and coincides with the wave operator as defined in (72). □

Remark 7.5. The operator $\overset{\mathrm{s}}{W}_\pm(H_r, H_0)$ acts on $\mathcal{H}$, while the operator $W_\pm(H_r, H_0)$ acts on the direct integral $\mathfrak{H}$. In Theorem 7.4 by $W_\pm(H_r, H_0)$ one, of course, means the operator

$$\mathcal{E}^*(H_r) W_\pm(H_r, H_0) \mathcal{E}(H_0) \colon \mathcal{H} \to \mathcal{H}.$$

Theorem 7.4, in particular, shows that the operators $W_\pm(H_r, H_0)$ are independent from the choice of the frame F in the sense that the operator $\mathcal{E}^*(H_r)W_\pm(H_r, H_0)\mathcal{E}(H_0)$ is independent from F.

8. The scattering matrix

In [Y] the scattering matrix $S(\lambda; H_1, H_0)$ is defined via a direct integral decomposition of the scattering operator $\mathbf{S}(H_1, H_0)$. In our approach, we first define $S(\lambda; H_1, H_0)$, while the scattering operator $\mathbf{S}(H_1, H_0)$ is defined as a direct integral of $S(\lambda; H_1, H_0)$.

Definition 8.1. *For $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$ we define the scattering matrix $S(\lambda; H_r, H_0)$ by the formula*

$$S(\lambda; H_r, H_0) := w_+^*(\lambda; H_r, H_0) w_-(\lambda; H_r, H_0). \tag{80}$$

We list some properties of the scattering matrix which immediately follow from this definition (cf. [Y, Chapter 7]).

Theorem 8.2. *Let $\{H_r\}$ be a path of operators which satisfy Assumption 5.1. Let $\lambda \in \Lambda(H_0; F)$ and $r \notin R(\lambda; \{H_r\}, F)$. The scattering matrix $S(\lambda; H_r, H_0)$ possesses the following properties.*

(i) $S(\lambda; H_r, H_0) \colon \mathfrak{h}_\lambda^{(0)} \to \mathfrak{h}_\lambda^{(0)}$ *is a unitary operator.*

(ii) *For any h such that $r + h \notin R(\lambda; \{H_r\}, F)$ the equality*

$$S(\lambda; H_{r+h}, H_0) = w_+^*(\lambda; H_r, H_0) S(\lambda; H_{r+h}, H_r) w_-(\lambda; H_r, H_0)$$

holds.

(iii) *For any h such that $r + h \notin R(\lambda; \{H_r\}, F)$ the equality*

$$S(\lambda; H_{r+h}, H_0) = w_+^*(\lambda; H_r, H_0) S(\lambda; H_{r+h}, H_r) w_+(\lambda; H_r, H_0) S(\lambda; H_r, H_0)$$

holds.

Proof. (i) By Corollary 6.17 the operators $w_+^*(\lambda; H_r, H_0)$ and $w_-(\lambda; H_r, H_0)$ are unitary. It follows that their product $S(\lambda; H_r, H_0) = w_+^*(\lambda; H_r, H_0) w_-(\lambda; H_r, H_0)$ is also unitary.
(ii) From the definition of the scattering matrix (80) and multiplicative property of the wave matrix (Theorem 6.16) it follows that

$$\begin{aligned} S(\lambda; H_{r+h}, H_0) &= w_+^*(\lambda; H_{r+h}, H_0) w_-(\lambda; H_{r+h}, H_0) \\ &= (w_+(\lambda; H_{r+h}, H_r) w_+(\lambda; H_r, H_0))^* w_-(\lambda; H_{r+h}, H_r) w_-(\lambda; H_r, H_0) \\ &= w_+(\lambda; H_r, H_0)^* w_+(\lambda; H_{r+h}, H_r)^* w_-(\lambda; H_{r+h}, H_r) w_-(\lambda; H_r, H_0) \\ &= w_+(\lambda; H_r, H_0)^* S(\lambda; H_{r+h}, H_r) w_-(\lambda; H_r, H_0). \end{aligned}$$

Note that since $r, r+h \notin R(\lambda; \{H_r\}, F)$, all the operators above make sense.
(iii) It follows from (ii) and unitarity of the wave matrix (Corollary 6.17), that

$$\begin{aligned} S(\lambda; H_{r+h}, H_0) &= w_+(\lambda; H_r, H_0)^* S(\lambda; H_{r+h}, H_r) w_-(\lambda; H_r, H_0) \\ &= w_+(\lambda; H_r, H_0)^* S(\lambda; H_{r+h}, H_r) w_+(\lambda; H_r, H_0)(w_+^*(\lambda; H_r, H_0) w_-(\lambda; H_r, H_0)) \\ &= w_+^*(\lambda; H_r, H_0) S(\lambda; H_{r+h}, H_r) w_+(\lambda; H_r, H_0) S(\lambda; H_r, H_0). \end{aligned}$$

The proof is complete. □

We define the scattering operator by the formula

$$\mathbf{S}(H_r, H_0) := \int^{\oplus}_{\Lambda(H_r;F)\cap\Lambda(H_0;F)} S(\lambda; H_r, H_0)\, d\lambda. \tag{81}$$

Note that the scattering operator thus defined does not depend on the frame operator F. It follows from the definition of the wave operator (72) and the definition of the scattering matrix that

$$\mathbf{S}(H_r, H_0) = W_+^*(H_r, H_0) W_-(H_r, H_0),$$

which is a usual definition of the scattering operator.

By Remark 7.5, the definition of the scattering operator (81) is independent from the choice of the frame operator F.

Theorem 8.3. [Y, Chapter 7] *The scattering operator (81) has the following properties*

(i) *The scattering operator $\mathbf{S}(H_r, H_0)\colon \mathcal{H}^{(a)}(H_0) \to \mathcal{H}^{(a)}(H_0)$ is unitary.*
(ii) *The equality*

$$\mathbf{S}(H_{r+h}, H_0) = W_+(H_0, H_r)\mathbf{S}(H_{r+h}, H_r) W_-(H_r, H_0)$$

holds.
(iii) *The equality*

$$\mathbf{S}(H_{r+h}, H_0) = W_+(H_0, H_r)\mathbf{S}(H_{r+h}, H_r) W_+(H_r, H_0)\mathbf{S}(H_r, H_0)$$

holds.
(iv) *The equality*

$$\mathbf{S}(H_r, H_0) H_0 = H_0 \mathbf{S}(H_r, H_0)$$

holds.

Proof. (i) This follows from Theorem 8.2(i).
(ii) This follows from Theorem 8.2(ii).
(iii) This follows from Theorem 8.2(iii).
(iv) follows from the definition of the scattering operator (80) and Theorem 4.18. □

8.1. **Stationary formula for the scattering matrix.** The aim of this subsection is to prove the stationary formula for the scattering matrix.

Lemma 8.4. *If $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$, then*

$$\begin{aligned}(82)\qquad &(1 + R_{\lambda - i0}(H_0)V_r) \cdot \operatorname{Im} R_{\lambda+i0}(H_r) \cdot (1 + V_r R_{\lambda-i0}(H_0)) \\ &\qquad = \operatorname{Im} R_{\lambda+i0}(H_0)\Big[(1 - 2iV_r[1 - R_{\lambda+i0}(H_r)V_r]) \operatorname{Im} R_{\lambda+i0}(H_0)\Big]\end{aligned}$$

as equality in $\mathcal{L}_1(\mathcal{H}_1, \mathcal{H}_{-1})$.

Proof. We write

$$R_0 = R_{\lambda+i0}(H_0), \quad R_0^* = R_{\lambda-i0}(H_0), \quad R_r = R_{\lambda+i0}(H_r), \quad R_r^* = R_{\lambda-i0}(H_r).$$

Then the last formula becomes

$$(83)\qquad (1 + R_0^* V_r) \cdot \operatorname{Im} R_r \cdot (1 + V_r R_0^*) = \operatorname{Im} R_0 \Big[1 - 2iV_r(1 - R_r V_r) \operatorname{Im} R_0\Big].$$

Note that by the second resolvent identity

$$(84)\qquad R_r = (1 - R_r V_r) R_0.$$

Using (61), one has

$$(1 + R_0^* V_r) \operatorname{Im} R_r = \operatorname{Im} R_0 (1 - V_r R_r).$$

Further, using (84),

$$\begin{aligned} 1 - 2iV_r(1 - R_r V_r) \operatorname{Im} R_0 &= 1 - V_r(1 - R_r V_r)(R_0 - R_0^*) \\ &= 1 - V_r(1 - R_r V_r)R_0 + V_r(1 - R_r V_r)R_0^* \\ &= 1 - V_r R_r + V_r(1 - R_r V_r) R_0^* \\ &= (1 - V_r R_r)(1 + V_r R_0^*). \end{aligned}$$

Combining the last two formulae completes the proof. □

In the following theorem, we establish for trace-class perturbations well-known stationary formula for the scattering matrix (cf. [Y, Theorems 5.5.3, 5.5.4, 5.7.1]).

Theorem 8.5. *For any $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$ the stationary formula for the scattering matrix*

$$(85)\qquad S(\lambda; H_r, H_0) = 1_\lambda - 2\pi i \mathcal{E}_\lambda(H_0) V_r (1 + R_{\lambda+i0}(H_0)V_r)^{-1} \mathcal{E}_\lambda^\diamondsuit(H_0).$$

holds.

(The meaning of notation 1_λ is clear, though the subscript λ will be often omitted).

Proof. For $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$, the second resolvent identity

$$R_z(H_r) - R_z(H_0) = -R_z(H_r)V_r R_z(H_0) = -R_z(H_0) V_r R_z(H_r),$$

implies that the stationary formula can be written as

$$(86)\qquad S(\lambda; H_r, H_0) = 1 - 2\pi i \mathcal{E}_\lambda(H_0) V_r (1 - R_{\lambda+i0}(H_r)V_r) \mathcal{E}_\lambda^\diamondsuit(H_0).$$

It follows that it is enough to prove the equality

$$w_+^*(\lambda; H_r, H_0) w_-(\lambda; H_r, H_0) = 1 - 2\pi i \mathcal{E}_\lambda(H_0) V_r (1 - R_{\lambda+i0}(H_r) V_r) \mathcal{E}_\lambda^\diamondsuit(H_0).$$

Since the set $\mathcal{E}_\lambda(H_0)\mathcal{H}_1$ is dense in $\mathfrak{h}_\lambda^{(0)} = \mathfrak{h}_\lambda(H_0)$, it is enough to show that for any $f, g \in \mathcal{H}_1$

$$\begin{aligned} &\left\langle \mathcal{E}_\lambda(H_0) f, w_+^*(\lambda; H_r, H_0) w_-(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) g \right\rangle_{\mathfrak{h}_\lambda^{(0)}} \\ &\qquad = \left\langle \mathcal{E}_\lambda f, \left(1 - 2\pi i \mathcal{E}_\lambda V_r (1 - R_{\lambda+i0}(H_r) V_r) \mathcal{E}_\lambda^\diamondsuit\right) \mathcal{E}_\lambda g \right\rangle_{\mathfrak{h}_\lambda^{(0)}}. \end{aligned}$$

In other words, using Lemma 8.4 and (62), it is enough to show that

$$\begin{aligned} (E) &:= \langle w_+(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) f, w_-(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) g \rangle_{\mathfrak{h}_\lambda^{(r)}} \\ &= \left\langle f, (1 + R_{\lambda-i0}(H_0) V_r) \frac{1}{\pi} \operatorname{Im} R_{\lambda+i0}(H_r) \left(1 + V_r R_{\lambda-i0}(H_0)\right) g \right\rangle_{1,-1}. \end{aligned}$$

Let $\varepsilon > 0$. Since $\mathcal{E}_\lambda(H_r)\mathcal{H}_1$ is dense in $\mathfrak{h}_\lambda^{(r)}$ (see (41)), there exists $h \in \mathcal{H}_1$ such that the vector

$$a := w_+(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) f - \mathcal{E}_\lambda(H_r) h \ \in \ \mathfrak{h}_\lambda^{(r)} \tag{87}$$

has norm less than ε. Definition (63) of $w_-(\lambda; H_r, H_0)$ implies that

$$\begin{aligned} (E) &= \langle w_+(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) f, w_-(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) g \rangle_{\mathfrak{h}_\lambda^{(r)}} \\ &= \langle \mathcal{E}_\lambda(H_r) h + a, w_-(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) g \rangle_{\mathfrak{h}_\lambda^{(r)}} \\ &= \langle h, \mathfrak{a}_-(\lambda; H_r, H_0)(\lambda) g \rangle_{1,-1} + \langle a, w_-(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) g \rangle_{\mathfrak{h}_\lambda^{(r)}}. \end{aligned}$$

So, by the second equality of (61)

$$(E) = \left\langle h, \frac{1}{\pi} \operatorname{Im} R_{\lambda+i0}(H_r) [1 + V_r R_{\lambda-i0}(H_0)] g \right\rangle + \langle a, w_-(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) g \rangle.$$

Further, by (62) and (87),

$$\begin{aligned} (E) &= \langle \mathcal{E}_\lambda(H_r) h, \mathcal{E}_\lambda(H_r) [1 + V_r R_{\lambda-i0}(H_0)] g \rangle \\ &\qquad + \langle a, w_-(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) g \rangle \\ &= \langle w_+(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) f - a, \mathcal{E}_\lambda(H_r) [1 + V_r R_{\lambda-i0}(H_0)] g \rangle \\ &\qquad + \langle a, w_-(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) g \rangle \\ &= \langle \mathcal{E}_\lambda(H_0) f, w_+(\lambda; H_0, H_r) \mathcal{E}_\lambda(H_r) [1 + V_r R_{\lambda-i0}(H_0)] g \rangle \\ &\qquad - \langle a, \mathcal{E}_\lambda(H_r) [1 + V_r R_{\lambda-i0}(H_0)] g \rangle + \langle a, w_-(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) g \rangle. \end{aligned}$$

By definition (63) of $w_+(\lambda; H_r, H_0)$, it follows that

$$(E) = \langle f, \mathfrak{a}_+(\lambda; H_0, H_r) [1 + V_r R_{\lambda-i0}(H_0)] g \rangle + \text{remainder},$$

where

$$\text{remainder} := \langle a, w_-(\lambda; H_r, H_0) \mathcal{E}_\lambda(H_0) g - \mathcal{E}_\lambda(H_r) [1 + V_r R_{\lambda-i0}(H_0)] g \rangle.$$

By the first equality of (61),

$$(E) = \left\langle f, [1 + R_{\lambda-i0}(H_0)V_r]\frac{1}{\pi}\operatorname{Im} R_{\lambda+i0}(H_r)[1 + V_r R_{\lambda-i0}(H_0)]g \right\rangle + \text{remainder}.$$

Since the norm of the remainder term can be made arbitrarily small, it follows that

$$(E) = \left\langle f, [1 + R_{\lambda-i0}(H_0)V_r]\frac{1}{\pi}\operatorname{Im} R_{\lambda+i0}(H_r)[1 + V_r R_{\lambda-i0}(H_0)]g \right\rangle.$$

The proof is complete. □

As it can be seen from the proof, the remainder term in the proof of the last theorem is actually equal to zero and so it does not depend on a choice of the vector $h \in \mathcal{H}_1$; that is, for any $f, g \in \mathcal{H}$

$$\langle w_+(\lambda; H_r, H_0)\mathcal{E}_\lambda(H_0)f, w_-(\lambda; H_r, H_0)\mathcal{E}_\lambda(H_0)g - \mathcal{E}_\lambda(H_r)[1 + V_r R_{\lambda-i0}(H_0)]g\rangle = 0.$$

Since the set $w_+(\lambda; H_r, H_0)\mathcal{E}_\lambda(H_0)\mathcal{H}_1$ is dense in $\mathfrak{h}_\lambda(H_r)$, it follows that

Corollary 8.6. *For any $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$ the following equality holds:*

$$w_-(\lambda; H_r, H_0)\mathcal{E}_\lambda(H_0) = \mathcal{E}_\lambda(H_r)[1 + V_r R_{\lambda-i0}(H_0)].$$

Analogues equality can be written for $w_+(\lambda; H_r, H_0)$, but we don't need either of them.

Corollary 8.7. *If $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$, then $S(\lambda; H_r, H_0) \in 1 + \mathcal{L}_1(\mathfrak{h}_\lambda^{(0)})$.*

Proof. Since $\mathcal{E}_\lambda^\diamondsuit \in \mathcal{L}_2(\mathfrak{h}_\lambda^{(0)}, \mathcal{H}_{-1})$, $V \in \mathcal{B}(\mathcal{H}_{-1}, \mathcal{H}_1)$, $R_{\lambda+i0}(H_0) \in \mathcal{L}_2(\mathcal{H}_1, \mathcal{H}_{-1})$ and $\mathcal{E}_\lambda \in \mathcal{L}_2(\mathcal{H}_1, \mathfrak{h}_\lambda^{(0)})$, this follows from (86). □

Physicists (see e.g. [T]) write the stationary formula in a form as it looks in (85). The stationary formula can be written in the form (cf. [Y])

$$S(\lambda; H_r, H_0) = 1 - 2\pi i Z(\lambda; G)(1 + J_r T_0(\lambda + i0))^{-1} J_r Z^*(\lambda; G),$$

where $Z(\lambda; G)f = \mathcal{E}_\lambda(G^* f)$. The necessity to write the stationary formula in this form comes from the fact that in the abstract scattering theory the analogue of $\mathcal{E}_\lambda$ is not explicitly defined, but it becomes well defined for a dense linear manifold of vectors f from $\mathcal{H}$ and for a.e. λ in the composition $Z(\lambda; G)f = \mathcal{E}_\lambda(G^* f)$ (cf. [Y, §5.4]). In the present theory, both factors $\mathcal{E}_\lambda$ and G^* make sense, and as a consequence, there is no need to consider the operator $Z(\lambda; G)$.

Proposition 8.8. *The scattering matrix $S(\lambda; H_r, H_0)$ is a meromorphic function of r with values in $1 + \mathcal{L}_1(\mathfrak{h}_\lambda^{(0)})$, which admits analytic continuation to all its real poles.*

Proof. Since T_0 is compact, the function

$$\mathbb{R} \ni r \mapsto S(\lambda; H_r, H_0) \in 1 + \mathcal{L}_1(\mathfrak{h}_\lambda^{(0)})$$

admits meromorphic continuation to $\mathbb{C}$ by (85) and the analytic Fredholm alternative (see Theorem 2.17). Since $S(\lambda; H_r, H_0)$ is also bounded (unitary-valued) on the set

$\{r \in \mathbb{R} \colon \lambda \in \Lambda(H_r;F)\}$, which by Theorem 5.8 has discrete complement in $\mathbb{R}$, it follows that $S(\lambda; H_r, H_0)$ has analytic continuation to $\mathbb{R} \subset \mathbb{C}^{(r)}$, that is, the Laurent expansion of $S(\lambda; H_r, H_0)$ (as a function of the coupling constant r) in a neighbourhood of any resonance point $r_0 \in R(\lambda; \{H_r\}, F)$ does not have negative powers of $r - r_0$. □

Though this proposition is quite straightforward it seems to be new (to the best knowledge of the author). Proposition 8.8 asserts that the scattering matrix does not notice, in a certain sense, resonance points. There is a modified "scattering matrix"

$$\widetilde{S}(\lambda + i0; H_r, H_0) = 1 - 2ir\sqrt{\operatorname{Im} T_0(\lambda + i0)}J(1 + rT_0(\lambda + i0)J)^{-1}\sqrt{\operatorname{Im} T_0(\lambda + i0)},$$

introduced in [Pu], which, unlike the scattering matrix, does notice the resonance points. This has some implications which have been discussed in [Az$_2$] and in the setting of this paper they will be discussed in [Az$_3$].

8.2. **Infinitesimal scattering matrix.** Let $\{H_r\}$ be a path of operators which satisfies Assumption 5.1.

If $\lambda \in \Lambda(H_0;F)$, then, by 3.14(vi), the Hilbert-Schmidt operator $\mathcal{E}_\lambda \colon \mathcal{H}_1 \to \mathfrak{h}_\lambda$ is well defined. Hence, for any $\lambda \in \Lambda(H_0;F)$, it is possible to introduce the *infinitesimal scattering matrix*

$$\Pi_{H_0}(\dot H_0)(\lambda) \colon \mathfrak{h}_\lambda^{(0)} \to \mathfrak{h}_\lambda^{(0)}$$

by the formula

$$\Pi_{H_0}(\dot H_0)(\lambda) = \mathcal{E}_\lambda(H_0)\dot H_0 \mathcal{E}_\lambda^\diamondsuit(H_0), \tag{88}$$

where $\mathcal{E}_\lambda^\diamondsuit \colon \mathfrak{h}_\lambda \to \mathcal{H}_{-1}$ is a Hilbert-Schmidt operator as well (see Subsection 3.5.1). Here by $\dot H_0$ we mean the value of the trace-class derivative $\dot H_r$ at $r = 0$. Since $\mathcal{E}_\lambda(H_0)$ and $\mathcal{E}_\lambda^\diamondsuit(H_0)$ are Hilbert-Schmidt operators, and $\dot H_0 \colon \mathcal{H}_{-1} \to \mathcal{H}_1$ is bounded, it follows that $\Pi_{H_0}(\dot H_0)(\lambda)$ is a self-adjoint trace-class operator on the fiber Hilbert space $\mathfrak{h}_\lambda^{(0)}$.

The notion of infinitesimal scattering matrix was introduced in [Az].

Lemma 8.9. *Let $\{H_r\}$ be a path as above. Let r_0 be a point of analyticity of H_r. If $\lambda \in \Lambda(H_{r_0};F)$, then $\lambda \in \Lambda(H_r;F)$ for all r close enough to r_0 and*

$$\frac{d}{dr} S(\lambda; H_r, H_{r_0})\big|_{r=r_0} = -2\pi i \Pi_{H_{r_0}}(\dot H_{r_0})(\lambda),$$

where the derivative is taken in $\mathcal{L}_1(\mathfrak{h}_\lambda^{(0)})$-topology.

Proof. By Theorem 5.8, if $\lambda \in \Lambda(H_{r_0};F)$, then $\lambda \in \Lambda(H_r;F)$ for all r from some neighbourhood of r_0. Without loss of generality we can assume that $r_0 = 0$. We have

$$\begin{aligned}\frac{d}{dr} V_r(1 + R_{\lambda+i0}(H_0)V_r)^{-1} &= \dot V_r(1 + R_{\lambda+i0}(H_0)V_r)^{-1} \\ &\quad - V_r(1 + R_{\lambda+i0}(H_0)V_r)^{-1}R_{\lambda+i0}(H_0)\dot V_r(1 + R_{\lambda+i0}(H_0)V_r)^{-1},\end{aligned} \tag{89}$$

where the derivative is taken in $\mathcal{L}_1(\mathcal{H}_{-1}, \mathcal{H}_1)$. Since $V_0 = 0$ and $\dot H_r = \dot V_r$, this and Theorem 8.5 imply that

$$\begin{aligned}
\frac{d}{dr} S(\lambda; H_r, H_0)\big|_{r=r_0} &= \frac{d}{dr}\left(1_\lambda - 2\pi i \mathcal{E}_\lambda(H_0) V_r (1 + R_{\lambda+i0}(H_0)V_r)^{-1} \mathcal{E}^\diamondsuit_\lambda(H_0)\right)\big|_{r=0} \\
&= -2\pi i \mathcal{E}_\lambda(H_0) \cdot \frac{d}{dr}\left(V_r(1 + R_{\lambda+i0}(H_0)V_r)^{-1}\right)\big|_{r=0} \cdot \mathcal{E}^\diamondsuit_\lambda(H_0) \\
&= -2\pi i \mathcal{E}_\lambda(H_0) \dot H(0) \mathcal{E}^\diamondsuit_\lambda(H_0).
\end{aligned} \tag{90}$$

This and (88) complete the proof. □

Theorem 8.10. *If* $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$, *then*

$$\frac{d}{dr} S(\lambda; H_r, H_0) = -2\pi i w_+(\lambda; H_0, H_r) \Pi_{H_r}(\dot H_r)(\lambda) w_+(\lambda; H_r, H_0) S(\lambda; H_r, H_0), \tag{91}$$

where the derivative is taken in the trace-class norm.

Proof. By Theorem 5.8, for all small enough h the inclusion $\lambda \in \Lambda(H_{r+h}; F)$ holds. It follows from Theorem 8.2(iii) and unitarity of $w_\pm(\lambda; H_r, H_0)$ (Corollary 6.17) that

$$\begin{aligned}
&S(\lambda; H_{r+h}, H_0) - S(\lambda; H_r, H_0) \\
&\qquad = w_+(\lambda; H_0, H_r)\Big[S(\lambda; H_{r+h}, H_r) - 1_\lambda\Big] w_+(\lambda; H_r, H_0) S(\lambda; H_r, H_0).
\end{aligned}$$

Dividing this equality by h and taking the trace-class limit $h \to 0$ in it we get

$$\begin{aligned}
&\frac{d}{dh} S(\lambda; H_{r+h}, H_0)\big|_{h=0} \\
&\qquad = w_+(\lambda; H_0, H_r) \frac{d}{dh} S(\lambda; H_{r+h}, H_r)\big|_{h=0} w_+(\lambda; H_r, H_0) S(\lambda; H_r, H_0).
\end{aligned}$$

So, Lemma 8.9 completes the proof. □

Definition of the chronological exponential Texp, used in the next theorem, is given in Appendix A.

Theorem 8.11. *Let* $\{H_r\}$ *be a path of operators which satisfies Assumption 5.1. If* $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$, *then*

$$S(\lambda; H_r, H_0) = \operatorname{Texp}\left(-2\pi i \int_0^r w_+(\lambda; H_0, H_s) \Pi_{H_s}(\dot H_s)(\lambda) w_+(\lambda; H_s, H_0)\, ds\right), \tag{92}$$

where the chronological exponential is taken in the trace-class norm.

Proof. By Theorem 5.8, by the definition (88) of the infinitesimal scattering matrix and by Proposition 6.8, the expression under the integral in (92) makes sense for all s except the discrete resonance set $R(\lambda; \{H_r\}, F)$. The derivative $\frac{d}{dr} S(\lambda; H_r, H_0)$ is $\mathcal{L}_1(\mathfrak{h}^{(0)}_\lambda)$-analytic

by Proposition 8.8 and (89). Since $\mathbb{R} \ni r \mapsto S(\lambda; H_r, H_0)$ is also $\mathcal{L}_1(\mathfrak{h}_\lambda^{(0)})$-analytic, by the formula (91) the function

(93)
$$r \mapsto w_+(\lambda; H_0, H_r)\Pi_{H_r}(\dot H_r)(\lambda)w_+(\lambda; H_r, H_0) = -\frac{1}{2\pi i}\Big[\frac{d}{dr}S(\lambda; H_r, H_0)\Big]S(\lambda; H_r, H_0)^{-1}$$

is also $\mathcal{L}_1(\mathfrak{h}_\lambda^{(0)})$-analytic. Hence, integration of the equation (91) by Lemma A.1 gives (92). $\square$

Corollary 8.12. *Let $\lambda \in \Lambda(H_0; F)$. The function*

$$\mathbb{R} \ni r \mapsto \operatorname{Tr}\left(\Pi_{H_r}(\dot H_r)(\lambda)\right)$$

is piecewise analytic.

Proof. This follows from (93), unitarity of the wave matrix $w_+(\lambda; H_0, H_r)$ (Corollary 6.17) and unitarity and analyticity of the scattering matrix $S(\lambda; H_r, H_0)$ as a function of r (Proposition 8.8). $\square$

The formula (92) has an obvious physical interpretation.

For small $r \approx 0$, one can write

$$S(\lambda; H_r, H_0) \approx 1 - 2\pi i \int_0^r w_+(\lambda; H_0, H_s)\Pi_{H_s}(\dot H_s)(\lambda)w_+(\lambda; H_s, H_0)\, ds.$$

This formula is reminiscent to Born's approximation (cf. e.g. [Y, T]).

One can introduce *infinitesimal scattering operator* by the formula

$$\mathbf{\Pi}_{H_0}(V) = \int_{\Lambda(H_0;F)}^{\oplus} \Pi_{H_0}(V)(\lambda)\, d\lambda.$$

It follows from Theorem 8.11, that

(94)
$$\mathbf{S}(H_r, H_0) = \operatorname{Texp}\left(-2\pi i \int_0^r W_+(H_0, H_s)\mathbf{\Pi}_{H_s}(\dot H_s)W_+(H_s, H_0)\, ds\right),$$

where both operators act on $\mathcal{H}^{(a)}(H_0)$.

One can similarly prove a "reverse-time" (reverse-coupling constant) ordered analogue of this formula, with the right chronological exponential $\overrightarrow{\exp}$ instead of the left one $\operatorname{Texp} = \overleftarrow{\exp}$, and with $W_-(H_0, H_s)$ instead of $W_+(H_0, H_s)$:

(95)
$$\mathbf{S}(H_r, H_0) = \overrightarrow{\exp}\left(-2\pi i \int_0^r W_-(H_0, H_s)\mathbf{\Pi}_{H_s}(\dot H_s)W_-(H_s, H_0)\, ds\right),$$

where the right chronological exponential $\overrightarrow{\exp}$ is defined by

$$\overrightarrow{\exp}\left(\frac{1}{i}\int_a^t A(s)\, ds\right) = 1 + \sum_{k=1}^{\infty} \frac{1}{i^k}\int_a^t dt_k \int_a^{t_k} dt_{k-1} \ldots \int_a^{t_2} dt_1 A(t_1)\ldots A(t_k).$$

9. Absolutely continuous and singular spectral shifts

9.1. **Infinitesimal spectral flow.** In this subsection we prove a theorem, which asserts that the trace of the infinitesimal scattering matrix is a density of the absolutely continuous part of the infinitesimal spectral flow. In the previous version of this paper, the proof of this theorem used a specific frame associated with the trace-class operator V. In the current version, the proof is frame-independent, up to a less restrictive condition, given below. We recall that if $A\colon \mathcal{H} \to \mathcal{K}$ and $B\colon \mathcal{K} \to \mathcal{H}$ are two bounded operators, such that AB and BA are trace-class operators in Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$ respectively, then

$$\mathrm{Tr}_{\mathcal{K}}(AB) = \mathrm{Tr}_{\mathcal{H}}(BA). \tag{96}$$

Proof. Let $\dim \mathcal{K} \leqslant \dim \mathcal{H}$ and let $U\colon \mathcal{K} \to \mathcal{H}$ be an isometry with initial space $\mathcal{K}$, so that $U^*U = 1_{\mathcal{K}}$. Let (φ_j) be an orthonormal basis in $\mathcal{K}$ and let (ψ_k) be an orthonormal basis of $\mathcal{H}$, obtained from $(U\varphi_j)$ by adding, if necessary, new elements. Then

$$\begin{aligned}\mathrm{Tr}_{\mathcal{K}}(AB) &= \sum_j \langle \varphi_j, AB\varphi_j\rangle = \sum_k{}^{*} \langle U^*\psi_k, ABU^*\psi_k\rangle \\ &= \sum_k \langle U^*\psi_k, ABU^*\psi_k\rangle = \mathrm{Tr}_{\mathcal{H}}(UABU^*) = \mathrm{Tr}_{\mathcal{H}}(BU^*UA) = \mathrm{Tr}_{\mathcal{H}}(BA),\end{aligned}$$

where $\sum^*$ means that the sum is taken not over all basis (ψ_k).

Let $\{H_r\}$ be a path of self-adjoint operators which satisfies Assumption 5.1. In addition to this assumption, from now on we assume that

$$\sum_{j,k=1}^{\infty} \kappa_j \kappa_k J^r_{jk} \quad \text{is absolutely convergent,} \tag{97}$$

where $V_r = F^* J_r F$, and $\left(J^r_{jk}\right)$ is the matrix of J_r in the basis (ψ_k), that is, $J^r_{jk} = \langle \psi_j, J_r \psi_k\rangle$.

Obviously, for a straight line path $H_r = H_0 + rV$, there exists a frame F such that this additional condition holds (the frame from Lemma 5.2 will do).

Remark 5. V. V. Peller constructed an example of a trace-class operator $A = (a_{ij})$ and a bounded operator $B = (b_{ij})$ on ℓ_2, such that the double series

$$\sum_{i,j=1}^{\infty} |a_{ij} b_{ij}|$$

diverges[4].

Lemma 9.1. *The double series*

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \kappa_j \kappa_k J_{jk} \langle \varphi_j(\lambda), \varphi_k(\lambda)\rangle$$

[4]Private communication

is absolutely convergent for a.e. $\lambda \in \Lambda(H_0; F)$ *and the function*

$$\lambda \mapsto \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \kappa_j \kappa_k \left| J_{jk} \left\langle \varphi_j(\lambda), \varphi_k(\lambda) \right\rangle \right|$$

is integrable.

Proof. It follows from the assumption (97) and Corollary 4.4 that it is enough to prove the following assertion.

If a non-negative series $\sum_{j=1}^{\infty} a_n$ is convergent (to A) and if a sequence of integrable function $f_1, f_2, \ldots$ is such that $\|f_j\|_{L^1} \leqslant 1$ for all $j = 1, 2, \ldots$, then the series

$$\sum_{j=1}^{\infty} a_j f_j$$

is absolutely convergent a.e. and its sum is integrable.

By the Beppo-Levi Monotone Convergence theorem, the series $g(x) := \sum_{j=1}^{\infty} a_j \left| f_j \right|(x)$ is convergent (so far, possibly to $+\infty$) a.e. Since

$$\int \sum_{j=1}^{N} a_j \left| f_j \right|(x)\, dx \leqslant A$$

for all N, the series $\sum_{j=1}^{\infty} a_j \left| f_j \right|$ is absolutely convergent a.e. and its sum $g(x)$ is integrable. Since

$$\sum_{j=1}^{N} a_j \left| f_j \right|(x) \leqslant g(x),$$

it follows that, by the Lebesgue Dominated Convergence theorem, the series above is absolutely convergent and its sum is integrable. □

Theorem 9.2. *Let* H_0 *be a self-adjoint operator on a Hilbert space with a frame* F*. Let* V *be a trace-class operator such that (50) and (97) hold. Then for any bounded measurable function* h *the equality*

$$\operatorname{Tr}(V h(H_0^{(a)})) = \int_{\Lambda(H_0;F)} h(\lambda) \operatorname{Tr}_{\mathfrak{h}_\lambda}(\Pi_{H_0}(V)(\lambda))\, d\lambda$$

holds.

Proof. Since V satisfies (50), it has the representation

$$V = F^* J F, \tag{98}$$

where $J \colon \mathcal{K} \to \mathcal{K}$ is a bounded self-adjoint operator (not necessarily invertible). We recall that the frame operator F is given by (20). Let (J_{jk}) be the matrix of J in the basis (ψ_j) (see (20)), i.e.

$$J\psi_j = \sum_{k=1}^{\infty} J_{jk} \psi_k. \tag{99}$$

Using (96) and (98), we have

$$\mathrm{Tr}_{\mathcal{H}}(Vh(H_0^{(a)})) = \mathrm{Tr}_{\mathcal{K}}(JFh(H_0^{(a)})F^*).$$

Calculation of the trace in the right hand side of this formula in the orthonormal basis (ψ_j) of $\mathcal{K}$, together with (99) and (21) give

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}}(Vh(H_0^{(a)})) &= \sum_{j=1}^{\infty} \left\langle JFh(H_0^{(a)})F^*\psi_j, \psi_j \right\rangle \\
&= \sum_{j=1}^{\infty} \left\langle h(H_0^{(a)})F^*\psi_j, F^*J\psi_j \right\rangle \\
&= \sum_{j=1}^{\infty} \left\langle h(H_0^{(a)})F^*\psi_j, F^* \sum_{k=1}^{\infty} J_{jk}\psi_k \right\rangle \\
&= \sum_{j=1}^{\infty} \kappa_j \left\langle h(H_0^{(a)})\varphi_j, \sum_{k=1}^{\infty} J_{jk}\kappa_k\varphi_k \right\rangle \\
&= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \kappa_j\kappa_k J_{jk} \left\langle h(H_0^{(a)})\varphi_j, \varphi_k \right\rangle .
\end{aligned}$$

This double sum is absolutely convergent by the assumption (97) and the estimate $\left|\left\langle h(H_0^{(a)})\varphi_j, \varphi_k \right\rangle\right| \leqslant |h|_\infty$.

Now, combining the last equality with Theorem 4.18 and Corollary 4.16 implies

$$\mathrm{Tr}_{\mathcal{H}}(Vh(H_0^{(a)})) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \kappa_j\kappa_k J_{jk} \int_{\Lambda(H_0;F)} h(\lambda) \left\langle \varphi_j(\lambda), \varphi_k(\lambda)\right\rangle_{\mathfrak{h}_\lambda} \, d\lambda.$$

It follows from Lemma 9.1, that the integral and summations in the last equality can be interchanged:

$$\mathrm{Tr}_{\mathcal{H}}(Vh(H_0^{(a)})) = \int_{\Lambda(H_0;F)} h(\lambda) \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \kappa_j\kappa_k J_{jk} \left\langle \varphi_j(\lambda), \varphi_k(\lambda)\right\rangle_{\mathfrak{h}_\lambda} \, d\lambda. \tag{100}$$

On the other hand, by (96) and (98), for any $\lambda \in \Lambda(H_0; F)$

$$\mathrm{Tr}_{\mathfrak{h}_\lambda}(\mathcal{E}_\lambda V \mathcal{E}_\lambda^\diamondsuit) = \mathrm{Tr}_{\mathcal{K}}(JF\mathcal{E}_\lambda^\diamondsuit \mathcal{E}_\lambda F^*).$$

Similarly, evaluation of the last trace in the orthonormal basis (ψ_j) of $\mathcal{K}$ gives

$$\begin{aligned}\operatorname{Tr}_{\mathfrak{h}_\lambda}(\Pi_{H_0}(V)(\lambda)) = \operatorname{Tr}_{\mathfrak{h}_\lambda}(\mathcal{E}_\lambda V \mathcal{E}_\lambda^\diamondsuit) &= \sum_{j=1}^\infty \left\langle \mathcal{E}_\lambda^\diamondsuit \mathcal{E}_\lambda F^* \psi_j, F^* J \psi_j \right\rangle_{-1,1} \\ &= \sum_{j=1}^\infty \left\langle \mathcal{E}_\lambda F^* \psi_j, \mathcal{E}_\lambda F^* J \psi_j \right\rangle_{\mathfrak{h}_\lambda} \\ &= \sum_{j=1}^\infty \kappa_j \left\langle \mathcal{E}_\lambda \varphi_j, \mathcal{E}_\lambda \sum_{k=1}^\infty J_{jk} \kappa_k \varphi_k \right\rangle_{\mathfrak{h}_\lambda} \\ &= \sum_{j=1}^\infty \sum_{k=1}^\infty \kappa_j \kappa_k J_{jk} \left\langle \mathcal{E}_\lambda \varphi_j, \mathcal{E}_\lambda \varphi_k \right\rangle_{\mathfrak{h}_\lambda}.\end{aligned}$$

(The last equality here holds, since $\sum_{k=1}^\infty$ converges absolutely). Combining this equality with (100) completes the proof.

$\square$

The *infinitesimal spectral flow* $\Phi_{H_0}(V)$ is a distribution on $\mathbb{R}$, defined by the formula

$$\Phi_{H_0}(V)(\varphi) = \operatorname{Tr}(V\varphi(H_0)).$$

This notion was introduced in [ACS] and developed in [AS, Az]. The terminology "infinitesimal spectral flow" is justified by the following classical formula from formal perturbation theory [LL, (38.6)]

$$E_n^{(1)} = V_{nn},$$

where $V_{nn} = \langle n|V|n\rangle$ is the matrix element of the perturbation V, $E_n^{(1)}$ denotes the first correction term for the n-th eigenvalue $E_n^{(0)}$ (corresponding to $|n\rangle$) of the unperturbed operator H_0 perturbed by V. If the support of φ contains only the eigenvalue $E_n^{(0)}$ and $\varphi\left(E_n^{(0)}\right) = 1$, then $\operatorname{Tr}(V\varphi(H_0)) = V_{nn}$. So, $\Phi_{H_0}(V)(\varphi)$ measures the shift of eigenvalues of H_0. Another justification is that, according to the Birman-Solomyak formula (1), the spectral shift function is the integral of infinitesimal spectral flow $\Phi_{H_r}(V)(\delta)$.

Remark 9.3. *From now on, for an absolutely continuous measure μ we denote its density by the same symbol. So, in $\mu(\varphi)$, $\varphi \in C_c(\mathbb{R})$, μ is a measure, while in $\mu(\lambda)$, $\lambda \in \mathbb{R}$, μ is a function.*

Actually, $\Phi_{H_0}(V)$ is a measure [AS]. So, one can introduce the absolutely continuous and singular parts of the infinitesimal spectral flow:

$$\Phi_{H_0}^{(a)}(V)(\varphi) = \operatorname{Tr}(V\varphi(H_0^{(a)})),$$

and

$$\Phi_{H_0}^{(s)}(V)(\varphi) = \operatorname{Tr}(V\varphi(H_0^{(s)})).$$

Recall that for every $\lambda \in \Lambda(H_0; F)$ and any $V \in \mathcal{A}(F)$, we have a trace class operator

$$\Pi_{H_0}(V)(\lambda) \colon \mathfrak{h}_\lambda \to \mathfrak{h}_\lambda.$$

We define the *standard density* function of the absolutely continuous infinitesimal spectral flow by the formula

$$\Phi^{(a)}_{H_0}(V)(\lambda) := \operatorname{Tr}(\Pi_{H_0}(V)(\lambda)) \quad \text{for all} \quad \lambda \in \Lambda(H_0; F), \tag{101}$$

where $V \in \mathcal{A}(F)$, and where, allowing a little abuse of notation[5], we denote the density of the infinitesimal spectral flow $\Phi^{(a)}_{H_0}(V)$ by the same symbol $\Phi^{(a)}_{H_0}(V)(\cdot)$. Since $\Phi^{(a)}_{H_0}(V)$ is absolutely continuous, the usage of this notation should not cause any problems. This terminology and notation are justified by Theorem 9.2.

The a.c. part of ISF $\Phi^{(a)}_{H_0}(\cdot)(\lambda)$ can be looked at as a one-form on the affine space of operators

$$H_0 + \mathcal{A}(F).$$

The standard density $\Phi^{(a)}_{H_0}(V)(\cdot)$ of the absolutely continuous part of the infinitesimal spectral flow may depend on a frame operator F. But as Theorem 9.2 shows, for any two frames the corresponding standard densities are equal a.e.

Corollary 9.4. *For any two frames F_1 and F_2 the standard densities of the absolutely continuous part of the infinitesimal spectral flow coincide a.e.*

By $\gamma(\{H_r\}\,; F)$ we denote the set of all pairs $(r, \lambda) \in \mathbb{R}^2$ such that $\lambda \in \Lambda(H_r; F)$.

Lemma 9.5. *The set $\gamma(\{H_r\}\,; F) \subset \mathbb{R}^2$ is Borel measurable and the function (see (101))*

$$\gamma(\{H_r\}\,; F) \ni (r, \lambda) \mapsto \Phi^{(a)}_{H_r}(\dot H_r)(\lambda) \tag{102}$$

is also measurable. Moreover, the complement of $\gamma(\{H_r\}\,; F)$ is a null set in $\mathbb{R}^2$.

Proof. The set $\gamma(\{H_r\}\,; F)$ is Borel measurable since it is the (intersection of two) set of points of convergence of a family of continuous functions

$$F R_z(H_r) F^*$$

of two variables r and $z = \lambda + iy$ (see Definition 3.2).

The function $(r, \lambda) \mapsto \Phi^{(a)}_{H_r}(\dot H_r)(\lambda)$ is measurable since

$$\begin{aligned}\Phi^{(a)}_{H_r}(\dot H_r)(\lambda) = \operatorname{Tr}(\Pi_{H_r}(\dot H_r)(\lambda)) &= \operatorname{Tr}\left(\mathcal{E}_\lambda(H_r)\dot H_r \mathcal{E}^\diamondsuit_\lambda(H_r)\right) \\ &= \lim_{y \to 0^+} \operatorname{Tr}\left(\mathcal{E}_{\lambda+iy}(H_r)\dot H_r \mathcal{E}^\diamondsuit_{\lambda+iy}(H_r)\right),\end{aligned}$$

where the last equality follows from the fact that $\mathcal{E}^\diamondsuit_{\lambda+iy}(H_r) \colon \mathfrak{h}_\lambda \to \mathcal{H}_{-1}$ is Hilbert-Schmidt (see subsection 3.14), $\dot H_r \colon \mathcal{H}_{-1} \to \mathcal{H}_1$ is bounded and $\mathcal{E}_{\lambda+iy}(H_r) \colon \mathcal{H}_1 \to \mathfrak{h}_\lambda$ is also Hilbert-Schmidt, and the operators $\mathcal{E}^\diamondsuit_{\lambda+iy}(H_r)$, $\mathcal{E}_{\lambda+iy}(H_r)$ converge to $\mathcal{E}^\diamondsuit_{\lambda+i0}(H_r)$,

[5] See Remark 9.3

$\mathcal{E}_{\lambda+i0}(H_r)$ in the Hilbert-Schmidt norm, so that the product $\mathcal{E}_{\lambda+iy}(H_r)\dot H_r\mathcal{E}^\diamondsuit_{\lambda+iy}(H_r)$ converges to $\mathcal{E}_\lambda(H_r)\dot H_r\mathcal{E}^\diamondsuit_\lambda(H_r)$ in the trace-class norm, as $y \to 0^+$.

That the complement of $\gamma(\{H_r\}\,;F)$ is a null set in $\mathbb{R}^2$ now follows from Fubini's Theorem, from the discreteness property of the resonance set with respect to r (Theorem 5.8) and from the fact that $\Lambda(H_r;F)$ is a full set (Proposition 3.3). □

Lemma 9.6. *Let $V \in \mathcal{A}(F)$. The function $\Lambda(H,F) \ni \lambda \mapsto \Phi^{(a)}_H(V)(\lambda)$ is summable.*

Proof. This function is a density of an absolutely continuous finite measure $\varphi \mapsto \Phi^{(a)}_H(V)(\varphi)$. □

9.2. **Absolutely continuous and singular spectral shifts.** Let $\{H_r\colon r \in [0,1]\}$ be a piecewise real-analytic path of operators.

We define the spectral shift function ξ, its absolutely continuous $\xi^{(a)}$ and singular $\xi^{(s)}$ parts as distributions by the formulae

$$\xi_{H_1,H_0}(\varphi) = \int_0^1 \Phi_{H_r}(\dot H_r)(\varphi)\,dr, \quad \varphi \in C_c^\infty(\mathbb{R}), \tag{103}$$

$$\xi^{(a)}_{H_1,H_0}(\varphi) = \int_0^1 \Phi^{(a)}_{H_r}(\dot H_r)(\varphi)\,dr, \quad \varphi \in C_c^\infty(\mathbb{R}), \tag{104}$$

$$\xi^{(s)}_{H_1,H_0}(\varphi) = \int_0^1 \Phi^{(s)}_{H_r}(\dot H_r)(\varphi)\,dr, \quad \varphi \in C_c^\infty(\mathbb{R}). \tag{105}$$

For the straight path $\{H_r = H_0 + rV\}$, the first of these formulae is the Birman-Solomyak spectral averaging formula [BS$_2$], which shows that the definition of the spectral shift function, given above, coincides with the classical definition of M. G. Kreĭn [Kr]. It will be shown later that these definitions do not depend on the choice of the path $\{H_r\}$ connecting H_0 and H_1.

Lemma 9.7. *The distribution $\xi^{(a)}_{H_1,H_0}$ is an absolutely continuous measure with density*[6]

$$\xi^{(a)}_{H_1,H_0}(\lambda) := \int_0^1 \Phi^{(a)}_{H_r}(\dot H_r)(\lambda)\,dr, \qquad \lambda \in \Lambda(H_0;F). \tag{106}$$

Proof. (A) The function $\Phi^{(a)}_{H_r}(\dot H_r)(\lambda)$ is summable on $[0,1]\times\mathbb{R}$. Indeed, by Lemma 9.5 this function is measurable and the L_1-norm of $\Phi^{(a)}_{H_r}(V)(\lambda)$ is uniformly bounded (by $\|V\|$) with respect to $r \in [0,1]$, as it follows from Theorem 9.2.

(B) It follows from (A) and Fubini's theorem, that for any bounded measurable function h in the iterated integral

$$\int_0^1 \int_{\mathbb{R}} h(\lambda)\Phi^{(a)}_{H_r}(\dot H_r)(\lambda)\,d\lambda\,dr$$

[6] See Remark 9.3

one can interchange the order of integrals. It follows from this and Theorem 9.2 that

$$\begin{aligned}
\xi^{(a)}_{H_1,H_0}(\varphi) &= \int_0^1 \Phi^{(a)}_{H_r}(\dot H_r)(\varphi)\,dr && \text{by (104)}\\
&= \int_0^1 \int_{\mathbb{R}} \Phi^{(a)}_{H_r}(\dot H_r)(\lambda)\varphi(\lambda)\,d\lambda\,dr && \text{by Thm. 9.2}\\
&= \int_{\mathbb{R}} \varphi(\lambda) \int_0^1 \Phi^{(a)}_{H_r}(\dot H_r)(\lambda)\,dr\,d\lambda\\
&= \int_{\mathbb{R}} \varphi(\lambda)\xi^{(a)}_{H_1,H_0}(\lambda)\,d\lambda.
\end{aligned}$$

□

In the last lemma we again denote by the same symbol $\xi^{(a)}_{H_1,H_0}$ an absolutely continuous measure and its density. We call the function $\xi^{(a)}_{H_1,H_0}(\lambda)$ the standard density of $\xi^{(a)}$. Note that $\xi^{(a)}_{H_r,H_0}$ is explicitly defined for all $\lambda \in \Lambda(H_0; F)$. It is not difficult to see that $\xi^{(a)}_{H_1,H_0}(\lambda)$, thus defined, coincides a.e. with the right hand side of the formula (3).

The definition (103) of the spectral shift function makes sense for all trace compatible perturbations V of H_0 [AS]. For all such perturbations and any smooth function $f \in C_c^\infty(\mathbb{R})$ the operator $f(H_0+V)-f(H_0)$ is trace-class, and the spectral shift function, defined by (103), satisfies the Krein trace formula (cf. [BS$_2$], [AS, Theorem 2.9])

$$\operatorname{Tr}(f(H_0+V)-f(H_0)) = \int_{-\infty}^{\infty} f'(\lambda)\xi(\lambda)\,d\lambda.$$

This justifies the definition (103).

By det we denote the classical Fredholm determinant (cf. e.g. [GK, S$_3$]). Since, by Corollary 8.7, $S(\lambda; H_r, H_0) \in 1 + \mathcal{L}_1(\mathfrak{h}^{(0)}_\lambda)$, the determinant $\det S(\lambda; H_r, H_0)$ makes sense.

Let $\lambda \in \Lambda(H_0; F)$. Note that, by Proposition 8.8, the function

$$\mathbb{R} \ni r \mapsto S(\lambda; H_r, H_0) \in 1 + \mathcal{L}_1(\mathfrak{h}^{(0)}_\lambda)$$

is continuous in $\mathcal{L}_1(\mathfrak{h}^{(0)}_\lambda)$. Hence, the function

$$\mathbb{R} \ni r \mapsto \det S(\lambda; H_r, H_0) \in \mathbb{T}$$

is also continuous, where $\mathbb{T} = \{z \in \mathbb{C}\colon\ |z| = 1\}$. So, it is possible to define a continuous function

$$\mathbb{R} \ni r \mapsto -\frac{1}{2\pi i}\log\det S(\lambda; H_r, H_0) \in \mathbb{R}$$

with zero value at 0.

Theorem 9.8. *For all $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$ the equality*

$$\xi^{(a)}_{H_r,H_0}(\lambda) = -\frac{1}{2\pi i}\log\det S(\lambda; H_r, H_0) \tag{107}$$

holds, where the logarithm is defined in such a way that the function

$$[0, r] \ni s \mapsto \log\det S(\lambda; H_s, H_0)$$

is continuous.

Proof. By definitions (106) and (101) of $\xi^{(a)}$ and $\Phi^{(a)}$ we have

$$\xi^{(a)}(\lambda; H_r, H_0) = \int_0^r \Phi^{(a)}_{H_s}(\dot H_r)(\lambda)\,ds = \int_0^r \mathrm{Tr}_{\mathfrak{h}^{(s)}_\lambda}(\Pi_{H_s}(\dot H_r)(\lambda))\,ds. \tag{108}$$

By Theorem 5.8 and by the definition (88) of the infinitesimal scattering matrix $\Pi_{H_s}(V)(\lambda)$, the integrand of the last integral is defined for all $s \in [0, r]$ except the *discrete* resonance set $R(\lambda; \{H_r\}, F)$, (see (54)). Moreover, by Corollary 8.12, the function

$$\mathbb{R} \ni s \mapsto \mathrm{Tr}_{\mathfrak{h}^{(s)}_\lambda}(\Pi_{H_s}(V)(\lambda))$$

is piecewise analytic. Consequently, the integral (108) is well defined.

Since, by Corollary 6.17, the operator $w_+(\lambda; H_s, H_0)\colon \mathfrak{h}^{(0)}_\lambda \to \mathfrak{h}^{(s)}_\lambda$ is unitary for all $s \notin R(\lambda; \{H_r\}, F)$, it follows from (108) that

$$\xi^{(a)}(\lambda; H_r, H_0) = \int_0^r \mathrm{Tr}_{\mathfrak{h}^{(0)}_\lambda}(w_+(\lambda; H_0, H_s)\Pi_{H_s}(V)(\lambda)w_+(\lambda; H_s, H_0))\,ds.$$

Theorem 8.11 and Lemma A.3 now imply

$$-2\pi i\xi^{(a)}(\lambda; H_r, H_0) = \log\det S(\lambda; H_r, H_0),$$

where the branch of the logarithm is chosen as in the statement of the theorem. □

Corollary 9.9. *If* $\lambda \in \Lambda(H_r; F) \cap \Lambda(H_0; F)$, *then*

$$e^{-2\pi i\xi^{(a)}(\lambda)} = \det S(\lambda; H_r, H_0).$$

Corollary 9.10. *The definitions (104) and (105) of the absolutely continuous and singular spectral shift functions do not depend on the choice of the path* $\{H_r\}$, *provided that it satisfies Assumption 5.1 and (97).*

Proof. Independence of (104) from the choice of the path follows directly from the formula (107) since the right hand side of it is path-independent. Path-independence of the singular spectral shift function $\xi^{(s)} = \xi - \xi^{(a)}$ follows from the path independence of ξ and $\xi^{(a)}$ (cf. [AS, Theorem 2.9]). □

Let $\xi^{(s)}(\lambda)$ (respectively, $\xi(\lambda)$) be the density of the absolutely continuous measure[7] $\xi^{(s)}(\varphi)$ (respectively, $\xi(\varphi)$). Since V is trace-class, Corollary 9.9 and the Birman-Krein formula (2)

$$e^{-2\pi i\xi(\lambda)} = \det S(\lambda; H_r, H_0) \quad \text{a.e. } \lambda \in \mathbb{R}$$

imply the following result.

[7] See Remark 9.3

Theorem 9.11. *Let H_0 be a self-adjoint operator and let V be a trace-class self-adjoint operator. The singular part $\xi^{(s)}(\lambda; H_0 + V, H_0)$ of the spectral shift function of the pair $(H_0, H_0 + V)$ is an a.e. integer-valued function.*

Proof. By Lemma 5.2, for the straight line path $\{H_r = H_0 + rV, r \in [0,1]\}$, which connects H_0 and $H_0 + V$, there exists a frame F, such that Assumption 5.1 and (97) hold. Consequently, combining Corollary 9.9 with the Birman-Krein formula

$$e^{-2\pi i \xi(\lambda)} = \det S(\lambda; H_r, H_0),$$

we get for a.e. $\lambda \in \mathbb{R}$

$$e^{-2\pi i \xi^{(a)}(\lambda)} = e^{-2\pi i \xi(\lambda)}.$$

It follows that $e^{-2\pi i \xi^{(s)}(\lambda)} = 1$ for a.e. λ, that is, $\xi^{(s)}(\lambda) \in \mathbb{Z}$ for a.e. λ. □

Theorem 9.11 suggests that the singular part of the spectral shift function measures the "spectral flow" of the singular spectrum regardless of its position with respect to absolutely continuous spectrum.

The path-independence of $\xi^{(a)}$ and $\xi^{(s)}$ (Corollary 9.10) imply

Theorem 9.12. *Let H_0 be a self-adjoint operator and let V_1 and V_2 be trace-class operators. If $V_1, V_2 \in \mathcal{A}(F)$ and the condition (97) holds for both V_1 and V_2, then*

$$\xi^{(a)}_{H_2,H_0} = \xi^{(a)}_{H_2,H_1} + \xi^{(a)}_{H_1,H_0} \quad \textit{and} \quad \xi^{(s)}_{H_2,H_0} = \xi^{(s)}_{H_2,H_1} + \xi^{(s)}_{H_1,H_0},$$

where $H_1 = H_0 + V_1$ and $H_2 = H_1 + V_2$.

Note that the space of trace class operators which satisfy conditions of this theorem is dense in $\mathcal{L}_1(\mathcal{H})$. This suggests that the additivity property of $\xi^{(a)}$ and $\xi^{(s)}$ must hold for all trace-class perturbations.

10. On alternative proof of integrity of $\xi^{(s)}$

Though Theorem 9.11 shows that $\xi^{(s)}(\lambda)$ is an a.e. integer-valued, it leaves a feeling of dissatisfaction, since the set of full measure, on which $\xi^{(s)}$ is defined, is not explicitly indicated.

Actually, it is possible to give another proof of the last theorem, which uses a natural decomposition of Pushnitski μ-invariant $\mu(\theta, \lambda)$ (cf. [Pu], cf. also [Az$_2$]) into absolutely continuous $\mu^{(a)}(\theta, \lambda)$ and singular $\mu^{(s)}(\theta, \lambda)$ parts, so that the Birman-Krein formula becomes a corollary of this result and Theorem 9.8, rather than the other way. In another paper [Az$_3$], it will be shown that $\mu^{(s)}(\theta, \lambda)$ does not depend on the angle variable θ and coincides with $-\xi^{(s)}(\lambda)$. Since the μ-invariant is integer-valued (it measures the spectral flow of partial scattering phase shifts), it follows that $\xi^{(s)}(\lambda)$ is integer-valued. The invariants $\mu(\theta, \lambda)$, $\mu^{(a)}(\theta, \lambda)$ and $\mu^{(s)}(\theta, \lambda)$ can be explicitly defined on $\Lambda(H_r; F) \cap \Lambda(H_0; F)$.

In this section I give definitions and formulate lemmas and theorems relevant to the second proof of Theorem 9.11. Proofs, which follow those in [Az$_2$], will appear in [Az$_3$].

10.1. **Absolutely continuous part of the Pushnitski μ-invariant.** Let $\lambda \in \Lambda(H_0; F)$. We denote by $e^{i\theta_j^*(r,\lambda)} \in \mathbb{T}$, $j = 1, 2, \ldots$ the eigenvalues of the scattering matrix $S(\lambda; H_r, H_0)$. Since, by Proposition 8.8, the scattering matrix $S(\lambda; H_r, H_0)$ is a meromorphic function, which is analytic for real r's, the arguments $\theta_j^*(\lambda, r)$ may and will be chosen to be continuous (real-analytic) functions of r, converging to 0 as $r \to 0$.

Definition 10.1. *The absolutely continuous part of the Pushnitski μ-invariant is the function*

$$[0, 2\pi) \times \Lambda(H_0; F) \ni (\theta, \lambda) \mapsto \mu^{(a)}(\theta, \lambda; H_r, H_0) = -\sum_{j=1}^{\infty} \Big[\frac{\theta - \theta_j^*(\lambda, r)}{2\pi}\Big]. \tag{109}$$

The sum on the right hand side measures the number of times eigenvalues $e^{i\theta_j^*(r,\lambda)}$ of $S(\lambda; H_r, H_0)$ cross the point $e^{i\theta} \in \mathbb{T}$ in counterclockwise direction as r moves away from 0. In other words, $\mu^{(a)}(\theta, \lambda; H_r, H_0)$ measures the spectral flow of the scattering phases $e^{i\theta_j^*(r,\lambda)}$.

It is not difficult to check that the series

$$\sum_{j=1}^{\infty} \theta_j^*(\lambda, r)$$

converges uniformly with respect to $r \in [0, 1]$.

Theorem 10.2. *For every $\lambda \in \Lambda(H_0; F)$ the equality*

$$\xi^{(a)}(\lambda; H_1, H_0) = -\frac{1}{2\pi} \int_0^{2\pi} \mu^{(a)}(\theta, \lambda; H_1, H_0)\, d\theta \tag{110}$$

holds.

Recall that $\xi^{(a)}(\lambda; H_1, H_0)$ is defined by the formula (106).

10.2. **Pushnitski μ-invariant.** Following [Pu, (4.1)], we define the M-function by the formula

$$M(z, r) = M(z; H_r, H_0) = (H_r - \bar z)\, R_z(H_r)\, (H_0 - z)\, R_{\bar z}(H_0), \tag{111}$$

and the $\tilde S$-function by the formula

$$\tilde S(z, r; H_0, G, J) = 1 - 2ir\sqrt{\operatorname{Im} T_0(z)}J(1 + rT_0(z)J)^{-1}\sqrt{\operatorname{Im} T_0(z)}, \tag{112}$$

where $\operatorname{Im} z > 0$. The M-function can be considered as a product of the Cayley transforms of operators H_r and H_0, and its values are unitary operators. It is not difficult to check that $\tilde S$ is also a unitary operator.

One can check that (see [Pu, (4.4)])

$$M(z; H_r, H_0) = 1 - 2iyR_z(H_r)V_rR_{\bar z}(H_0).$$

This equality, the estimate $\|R_z(H)\| \leqslant \frac{1}{|\operatorname{Im} z|}$ and the norm continuity of the function $\mathbb{C}_+ \times \mathbb{R} \ni (z, r) \mapsto R_z(H_r)$, imply the following two lemmas.

Lemma 10.3. *The function*

$$(z,r) \in \mathbb{C}_+ \times \mathbb{R} \mapsto M(z,r)$$

takes values in $\in 1 + \mathcal{L}_1(\mathcal{H})$ *and it is continuous in* $\mathcal{L}_1(\mathcal{H})$*-norm.*

Lemma 10.4. *When* $y \to +\infty$

$$\|M(\lambda + iy, H_r, H_0) - 1\|_1 \to 0$$

locally uniformly with respect to $r \in \mathbb{R}$.

By Lemma 5.4, the function $\tilde{S}(z,r)$ is also $\mathcal{L}_1$-continuous on $\mathbb{C}_+ \times \mathbb{R}$.

Proposition 10.5. [Pu, Theorem 4.1] *Eigenvalues of* $M(z; H_r, H_0)$ *and* $\widetilde{S}(z, r; H_0, G, J)$ *coincide (counting multiplicities).*

We denote by $e^{i\theta_j(z,r)}$ the eigenvalues of $M(z,r)$ $(= \tilde{S}(z,r))$ (counting multiplicities). We choose them in such a way, that the functions $\theta_j(z,r)$ are continuous in $\mathbb{C}_+ \times \mathbb{R}$ and $\theta_j(\lambda + iy, r) \to 0$, as $y \to +\infty$.

Proposition 10.6. *If* $\lambda \in \Lambda(H_0, H_r, G)$, *then the limit values* $e^{i\theta_j(\lambda+i0,r)}$ *of the eigenvalues of the* M*-function exist.*

Since the function $\mathbb{C} \ni r \mapsto \tilde{S}(\lambda + i0, r)$ is meromorphic, it follows that the limit functions $\theta_j(\lambda + i0, r)$ are continuous (actually, also meromorphic) outside the resonance set. At the same time the eigenvalues of $\tilde{S}(\lambda + i0, r)$ coincide with eigenvalues of the scattering matrix $S(\lambda; H_r, H_0)$.

Lemma 10.7. $\tilde{S}(\lambda + iy, r)$ *converges to* $\tilde{S}(\lambda + i0, r)$ *in* $\mathcal{L}_1(\mathcal{H})$ *locally uniformly outside of the resonance set as* $y \to 0$.

Lemma 10.8. *The arguments of the eigenvalues* $\theta_j(\lambda + iy, r)$ *converge locally uniformly outside of the resonance set as* $y \to 0$.

The Pushnitski μ-invariant is defined similarly, but instead of θ_j^*'s one takes θ_j's.

Definition 10.9. *Pushnitski* μ*-invariant is the function*

$$[0, 2\pi) \times \Lambda(H_0; F) \ni (\lambda, \theta) \mapsto \mu(\theta, \lambda; H_r, H_0) = -\sum_{j=1}^{\infty} \Big[\frac{\theta - \theta_j(\lambda + i0, r)}{2\pi}\Big].$$

The spectral shift distribution $\xi = \xi(\varphi)$ is an absolutely continuous measure. We denote by $\xi = \xi(\lambda)$ a density of this absolutely continuous measure.

Theorem 10.10. *For almost every* $\lambda \in \Lambda(H_0; F)$ *the equality*

$$\xi(\lambda; H_1, H_0) = -\frac{1}{2\pi} \int_0^{2\pi} \mu(\theta, \lambda; H_1, H_0)\, d\theta$$

holds.

This theorem allows to define explicitly the spectral shift function on the full set $\Lambda(H_0; F)$.

Definition 10.11. *Let $\lambda \in \Lambda(H_0; F)$. The Lifshits-Krein spectral shift function $\xi(\lambda)$ is by definition*

$$\xi(\lambda; H_1, H_0) = -\frac{1}{2\pi} \int_0^{2\pi} \mu(\theta, \lambda; H_1, H_0)\, d\theta.$$

The advantage of this definition of the spectral shift function is that it gives explicit values of ξ on an explicit set of full Lebesgue measure.

Definition 10.12. *The singular part of Pushnitski μ-invariant is the function*

$$\mu^{(s)}(\theta; \lambda) := \mu(\theta; \lambda) - \mu^{(a)}(\theta; \lambda).$$

Theorem 10.13. *The singular part of the Pushnitski μ-invariant $\mu^{(s)}(\theta, \lambda)$ does not depend on the angle variable θ. Thus defined function of the variable λ is minus the density $\xi^{(s)}$ of the singular part of the spectral shift function:*

$$\xi^{(s)}(\lambda) = -\mu^{(s)}(\lambda).$$

Consequently, the singular part of the spectral shift function $\xi^{(s)}(\lambda)$ is integer-valued.

Combined with Theorem 9.11 the last theorem gives a proof of

Theorem 10.14. (Birman-Kreĭn formula) *Let H_0 be a self-adjoint operator and V be a trace-class self-adjoint operator. Then for a.e. $\lambda \in \mathbb{R}$*

$$e^{-2\pi i \xi(\lambda; H_1, H_0)} = \det S(\lambda; H_1, H_0),$$

where $H_1 = H_0 + V$.

Remark 10.15. Note that the proof of the Birman-Krein formula, given here, does not use the rank-one perturbation argument. Thus, it gives a solution of the long-standing problem posed by M. Sh. Birman and D. R. Yafaev in their review papers [BY, BY$_2$].

11. Open questions

11.1. **Integrity property of $\xi^{(s)}$ in the case of trace compatible operators.** I recall the notion of trace compatible operators [AS]. One-dimensional affine space of self-adjoint operators $\mathcal{A} = \{H_0 + rV : r \in \mathbb{R}\}$ is called trace compatible, if for any compactly supported continuous function φ the operator

$$V\varphi(H_r) \tag{113}$$

is trace-class and the map

$$(V_1, V_2) \mapsto V_1 \varphi(H_0 + V_2)$$

is continuous from $\mathcal{A}^2$ to $\mathcal{L}_1(\mathcal{H})$. The condition (113) goes back to M. Sh. Birman.

One can see that definitions (103), (104) and (105) of ξ, $\xi^{(a)}$ and $\xi^{(s)}$ respectively, make sense for trace compatible operators H_0 and $H_0 + V$. It also can be seen that Birman's condition is the most general possible condition under which the definitions make sense.

A natural problem is to prove the integer-valuedness of the singular spectral shift function for trace compatible operators.

11.2. **Direct proof of integrity of the singular spectral shift.** Definitions of the absolutely continuous and singular spectral shift functions do not involve any notions of scattering theory at all. A natural question is to find a proof of integrity of the singular spectral shift function which does not use scattering theory.

11.3. **On examples with non-trivial singular spectral shift function.** It is not difficult to present examples of pairs (H_0, V) with $\xi^{(a)} \neq 0$ and $\xi^{(s)} \neq 0$, such that the intersection of Borel supports of $\xi^{(a)}$ and $\xi^{(s)}$ is not a null set. But there are no known examples of irreducible pairs with this property.

I recall some definitions. By a non-trivial subspace $\mathcal{K}$ of a Hilbert space $\mathcal{H}$ we mean a non-zero subspace of $\mathcal{H}$, not equal to $\mathcal{H}$; that is, $\{0\} \neq \mathcal{K} \neq \mathcal{H}$. A subspace $\mathcal{K}$ is invariant with respect to an operator H, if $H\mathcal{K} \subset \mathcal{K}$.

Definition 11.1. *A pair H_0 and V of two self-adjoint operators on a Hilbert space $\mathcal{H}$ is irreducible, if there does not exist a non-trivial subspace $\mathcal{K}$ of $\mathcal{H}$ which is invariant with respect to both operators H_0 and V.*

Problem 11.1. *Find an irreducible pair (if it exists) of a self-adjoint operator H_0 and a trace-class self-adjoint operator V such that the intersection of the sets*

$$\mathcal{A}(\xi^{(a)}_{H_0,H_0+V}(\cdot)) \quad \text{and} \quad \mathcal{A}(\xi^{(s)}_{H_0,H_0+V}(\cdot))$$

has non-zero Lebesgue measure.

Obviously, without irreducibility condition this problem has a trivial (positive) solution.

It should be possible to construct such examples using the path independence of $\xi^{(a)}$ and $\xi^{(s)}$.

11.4. **Pure point and singular continuous spectral shift functions.** One can also define and consider singular continuous $\xi^{(sc)}$ and pure point $\xi^{(pp)}$ spectral shift functions by formulas

$$\xi^{(sc)}(\varphi) = \int_0^1 \Phi^{(sc)}_{H_r}(\dot H_r)(\varphi)\,dr, \quad \varphi \in C_c^\infty(\mathbb{R}),$$

and

$$\xi^{(pp)}(\varphi) = \int_0^1 \Phi^{(pp)}_{H_r}(\dot H_r)(\varphi)\,dr, \quad \varphi \in C_c^\infty(\mathbb{R}),$$

where

$$\Phi^{(sc)}_H(V)(\varphi) = \operatorname{Tr}(V\varphi(H^{(sc)})) \quad \text{and} \quad \Phi^{(pp)}_H(V)(\varphi) = \operatorname{Tr}(V\varphi(H^{(pp)})).$$

Clearly,

$$\xi^{(s)} = \xi^{(sc)} + \xi^{(pp)}.$$

While $\xi^{(s)}$ is path-independent, I believe that $\xi^{(pp)}$ is path-dependent. So, in the above formulas we have to take $H_r = H_0 + rV$.

The generic property of the singular continuous spectrum (see [JS, RJLS, RJMS, RMS]) suggests that

Conjecture. Let V be trace class. If the pair (H_0, V) is irreducible, then $\xi^{(pp)}_{H_0+V,H_0} = 0$ on the absolutely continuous spectrum of H_0.

If this conjecture is true, it partly explains why it is difficult to construct explicit paths with non-trivial singular spectral shift function.

At the same time, it is possible that for non-trace-class perturbations V the function $\xi^{(pp)}_{H_0+V,H_0}$ can be non-zero on the absolutely continuous spectrum of H_0.

Appendix A. Chronological exponential

In this appendix an exposition of the chronological exponential is given. See e.g. [AgG, G] and [BSh, Chapter 4].

Let $p \in [1, \infty]$ and let $a < b$. Let $A(\cdot)\colon [a, b] \to \mathcal{L}_p(\mathcal{H})$ be a piecewise continuous path of self-adjoint operators from $\mathcal{L}_p(\mathcal{H})$. Consider the equation

$$\frac{dX(t)}{dt} = \frac{1}{i} A(t)X(t), \quad X(a) = 1, \tag{114}$$

where the derivative is taken in $\mathcal{L}_p(\mathcal{H})$. Let $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_k \leqslant t$. By definition, the left chronological exponent $\mathrm{Texp} = \overleftarrow{\exp}$ is

$$\mathrm{Texp}\left(\frac{1}{i}\int_a^t A(s)\,ds\right) = 1 + \sum_{k=1}^{\infty} \frac{1}{i^k} \int_a^t dt_k \int_a^{t_k} dt_{k-1} \ldots \int_a^{t_2} dt_1 A(t_k) \ldots A(t_1), \tag{115}$$

where the series converges in $\mathcal{L}_p(\mathcal{H})$-norm.

Lemma A.1. *The equation (114) has a unique continuous solution $X(t)$, given by formula*

$$X(t) = \mathrm{Texp}\left(\frac{1}{i}\int_a^t A(s)\,ds\right).$$

Proof. Substitution shows that (115) is a continuous solution of (114). Let $Y(t)$ be another continuous solution of (114). Taking the integral of (114) in $\mathcal{L}_p(\mathcal{H})$, one gets

$$Y(t) = 1 + \frac{1}{i}\int_a^t A(s)Y(s)\,ds.$$

Iteration of this integral and the bound $\sup_{t\in[a,b]} \|A(t)\|_p \leqslant \mathrm{const}$ show that $Y(t)$ coincides with (115). $\square$

A similar argument shows that $\mathrm{Texp}\left(\frac{1}{i}\int_a^t A(s)\,ds\right)X_0$ is the unique solution of the equation

$$\frac{dX(t)}{dt} = \frac{1}{i}A(t)X(t), \quad X(a) = X_0 \in 1 + \mathcal{L}_p(\mathcal{H}).$$

Lemma A.2. *The following equality holds*

$$\mathrm{Texp}\left(\int_s^u A(s)\,ds\right) = \mathrm{Texp}\left(\int_t^u A(s)\,ds\right)\mathrm{Texp}\left(\int_s^t A(s)\,ds\right).$$

Proof. Using (115), it is easy to check that both sides of this equality are solutions of the equation (in $\mathcal{L}_p(\mathcal{H})$)

$$\frac{dX(u)}{du} = \frac{1}{i}A(u)X(u)$$

with the initial condition $X(t) = \mathrm{Texp}\left(\int_s^t A(s)\,ds\right)$. So, Lemma A.1 completes the proof. $\square$

By det we denote the classical Fredholm determinant (cf. e.g. [GK, S_3, Y]).

Lemma A.3. *If $p = 1$ then the following equality holds*

$$\det\,\mathrm{Texp}\left(\frac{1}{i}\int_a^t A(s)\,ds\right) = \exp\left(\frac{1}{i}\int_a^t \mathrm{Tr}(A(s))\,ds\right).$$

Proof. Let $F(t)$ and $G(t)$ be the left and the right hand sides of this equality respectively. Then $\frac{d}{dt}G(t) = \frac{1}{i}\,\mathrm{Tr}(A(t))G(t)$, $G(a) = 1$. Further, by Lemma A.2 and the product property of det

$$\frac{d}{dt}F(t) = \lim_{h\to 0}\frac{1}{h}\left(\det\,\mathrm{Texp}\left(\frac{1}{i}\int_t^{t+h} A(s)\,ds\right) - 1\right)F(t) = \frac{1}{i}\,\mathrm{Tr}(A(t))F(t),$$

where the last equality follows from definitions of determinant [S_3, (3.5)], Texp and piecewise continuity of $A(s)$. $\square$

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